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Abstract

Full Text

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ON THE CONTINUATION OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

(Presented by Academician I. G. Petrovskii on 2 IV 1962)

1. Let D be a closed domain of n -dimensional space and let σ be a part of the boundary of the domain D . Let u be a solution of the linear differential equation $L(u) = 0$ in the domain D . Suppose that the coefficients of the operator L are also given in some neighborhood of σ . The question is whether the function u can be continued into this neighborhood so that the equation $L(u) = 0$ is satisfied.

Classical examples in these questions are the theorems of P. Courant ⁽¹⁾ on the continuation of solutions of Laplace's equation and of the equation $\Delta u + \lambda u = 0$. Among later results we mention the results of F. John ⁽²⁾ and G. Lewy ⁽³⁾. In the latter work it is shown that a solution of an elliptic equation with two independent variables and analytic coefficients, satisfying on a segment of the line $y = 0$ an analytic boundary condition, can be continued across this segment.

2. In the present note we show that, in some cases, the apparatus of transformation operators can be applied to the problem of continuation of solutions.

Denote by $E_h[0, b)$ the space of functions $f(x)$, twice differentiable on the interval $[0, b)$ and satisfying the boundary condition

$$f'(0) - hf(0) = 0.$$

Let

$$A(f) = \frac{d^2 f}{dx^2} - q(x)f, \quad B(f) = \frac{d^2 f}{dx^2} - r(x)f,$$

where $q(x)$ and $r(x)$ are continuous on the interval $[0, b)$.

The operator $X = X_{h_1, h_2; A, B}$ is called a **transformation operator** if it maps E_{h_1} into E_{h_2} and satisfies the conditions:

- 1)
$$AX = XB; \tag{1}$$

- 2) There exists a continuous inverse operator X^{-1} .

Fig. 1

Figure 1: Fig. 1

It is known that the operator X can be specified in the form

$$X(f) = f(x) + \int_0^x K(x, t)f(t) dt.$$

From condition (1) for the function $K(x, t)$ (the kernel of the transformation operator) it easily follows that:

$$\frac{\partial^2 K}{\partial x^2} - q(x)K = \frac{\partial^2 K}{\partial t^2} - r(t)K; \quad (2)$$

$$K(x, x) = h_2 - h_1 + \frac{1}{2} \int_0^x [q(s) - r(s)] ds; \quad (3)$$

$$\left[\frac{\partial K}{\partial t} - h_1 K \right]_{t=0} = 0. \quad (4)$$

3. Let us now consider the application of transformation operators to the continuation of solutions of differential equations. In order to explain our basic idea in the best way, we first consider one of the simplest cases.

Let D^+ be a convex domain in the (x, y) -plane whose boundary contains a rectilinear segment σ (see Fig. 1). Suppose that in the domain D^+ the function $u = u(x, y)$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - q(y)u = 0, \quad (5)$$

where $q(y)$ is a continuous function, and suppose that on the segment σ the function u satisfies the boundary condition

$$\left(\frac{\partial u}{\partial y} - hu \right) \Big|_{y=0} = 0, \quad (6)$$

where h is a constant number.

Fig. 1

Suppose that the coefficient $q(y)$ is extended continuously to the symmetric domain D^- . Put

$$A = \frac{d^2}{dy^2} - q(-y), \quad B = \frac{d^2}{dy^2} - q(y) \quad (y \geq 0).$$

Denote by u^+ the solution of equation (5) in the domain D^+ , and set

$$u^- = u(x, -y) = X(u^+) = u(x, y) + \int_0^y K(y, t)u(x, t) dt. \quad (7)$$

We shall show that u^- satisfies equation (5) in the domain D^- . Using (1) and equation (5), we have

$$A(u^-) = AX(u^+) = XB(u^+) = X\left(-\frac{\partial^2 u^+}{\partial x^2}\right) = -\frac{\partial^2}{\partial x^2}X(u^+) = -\frac{\partial^2 u^-}{\partial x^2},$$

which coincides with equation (5) (for u^-).

We now show that the continuation is continuous together with the first two derivatives. For u^- this follows directly from (7), if one sets $y = +0$ in this formula. Next, differentiating formula (7) with respect to y , we obtain

$$-u_y^- = u_y^+ + K(y, y)u(x, y) + \int_0^y \frac{\partial K}{\partial y} u(x, t) dt.$$

Setting here $y = +0$, we obtain, using the equality $K(0, 0) = -2h$ (which follows from equality (3) for $x = 0$),

$$\begin{aligned} -u_y(x, -0) &= u_y(x, +0) + K(0, 0)u(x, 0) \\ &= [h + K(0, 0)]u(x, 0) = -hu(x, 0) = -u_y(x, +0). \end{aligned}$$

Finally, setting in the equality

$$A(u^-) = XB(u^+)$$

$y = +0$, we obtain (using the continuity of $q(y)$ at zero)

$$u_{yy}(x, -0) - q(0)u(x, 0) = u_{yy}(x, +0) - q(0)u(x, 0).$$

Hence the continuity of the derivative u_{yy} follows.

If $q(y) = 0$, $h \neq 0$, then problem (2)–(3)–(4) is solved explicitly:

$$K(y, t) = -2he^{-h(y-t)}.$$

Therefore

$$u(x, -y) = u(x, y) - 2h \int_0^y e^{-h(y-t)} u(x, t) dt.$$

In the case of the boundary condition

$$u|_{y=0} = 0$$

the operator X should be taken in the form

$$X(f) = -f(y) + \int_0^y K(y, t) f(t) dt.$$

4. Let us now consider an equation of a more general form

$$a(x, y) \frac{\partial^2 u}{\partial y^2} + b(x, y) \frac{\partial u}{\partial y} + c(x, y)u + \alpha(x) \frac{\partial^2 u}{\partial x^2} + \beta(x) \frac{\partial u}{\partial x} + f(x, y) = 0, \quad (8)$$

in which $a(x, y)$ is a twice continuously differentiable function not vanishing anywhere; $b(x, y)$ has continuous first derivatives; $\alpha(x)$, $\beta(x)$, $f(x, y)$ are continuous functions in the domain $D^+ + \sigma$. Put $(y \geq 0)$

$$A = a(x, -y) \frac{\partial^2}{\partial y^2} - b(x, -y) \frac{\partial}{\partial y} + c(x, -y),$$

$$B = a(x, y) \frac{\partial^2}{\partial y^2} + b(x, y) \frac{\partial}{\partial y} + c(x, y).$$

In the boundary condition (6), h may now be either a number or a function of x .

Consider the transformation operator $X = X_{h, -h; A, B}$, which, obviously, now depends on x . It can be shown that the operator X is given by the formula

$$X(f) = \gamma(x, y) f[\delta(x, y)] + \int_0^{\delta(x, y)} K(x; y, t) f(t) dt,$$

where the functions $\gamma(x, y)$, $\delta(x, y)$, $K(x; y, t)$ are constructed from the operators A and B and the function $h = h(x)$.

Put, as before,

$$u^- = u(x, -y) = X(u^+).$$

From equation (8) it follows that

$$A(u^-) = AX(u^+) = XB(u^+) = -\alpha(x)X\left(\frac{\partial^2 u^+}{\partial x^2}\right) - \beta(x)X\left(\frac{\partial u^+}{\partial x}\right) - f^-, \quad (9)$$

where

$$f^- = f(x, -y) = X(f^+) \quad (f^+ = f(x, y), \quad y \geq 0).$$

Since the operator X now depends on x , it does not commute with $\partial/\partial x$. Therefore (9) is an integro-differential equation for u^- .

We note that in this case the partial derivatives $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial x \partial y$, upon passage across σ , undergo discontinuities of the first kind.

5. Our method is also applicable to hyperbolic equations. Consider, for example, in the domain D^+ , the mixed problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - q(y)u = \frac{\partial^2 u}{\partial t^2}; \quad (10)$$

$$\left(\frac{\partial u}{\partial y} - hu\right)\Big|_{y=0} = 0; \quad u|_{t=0} = f(x, y), \quad \frac{\partial u}{\partial t}\Big|_{t=0} = 0. \quad (11)$$

(h is a constant number.)

If we set

$$u^- = u(x, -y; t) = X(u^+) = u(x, y; t) + \int_0^y K(y, s) u(x, s; t) ds, \quad (12)$$

then it is easy to verify that u^- satisfies equation (10) in the domain D^- . Setting $t = 0$ in formula (12), we obtain a continuation for the initial function $f(x, y)$. If this function belongs to the class C^2 in the domain D^+ and satisfies the boundary condition (11), then its continuation will be of class C^2 in the domain $D^+ + \sigma + D^-$.

Thus, discarding the boundary condition (11), one may solve, in some neighborhood of the interval σ , the pure Cauchy problem (using the constructed continuation of the initial function). The solution of the pure Cauchy problem can be expressed in terms of the initial function $f(x, y)$ by known formulas. Using these formulas and the methods developed by us in papers (^{4, 5}), one can

study the asymptotic behavior of the spectral function of the equation all the way up to the boundary σ .

In conclusion, we note that the results presented here are generalized in an obvious way to the case of many variables.

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REFERENCES

1. R. Courant, D. Hilbert, *Methods of Mathematical Physics*, 1, 1951, p. 333.
2. F. Jon, Prolongement et reflexion des solutions des équations aux dérivées partielles, Colloques internationaux, Nancy, 1956.
3. H. Lewy, Bull. Am. Math. Soc., **65**, 37 (1959).
4. B. M. Levitan, Izv. AN SSSR, ser. matem., **20**, 437 (1956).
5. B. M. Levitan, Tr. Mosk. matem. obshch., **5**, 269 (1956).

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