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Abstract

Full Text

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ON AVERAGING ON AN UNBOUNDED INTERVAL

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In ([1], [2]) an asymptotic method was proposed for systems of the form

$$\dot{x} = \varepsilon X(x, y, t, \varepsilon), \qquad \dot{y} = Y(x, y, t, \varepsilon)$$

$(x = \{x_1, \dots, x_n\},\ y = \{y_1, \dots, y_m\},\ X = \{X_1, \dots, X_n\},\ Y = \{Y_1, \dots, Y_m\}, \varepsilon > 0$ is a small parameter),

connected with averaging along the integral curves of the degenerate system $x = \text{const}$, $\dot{y} = Y_0(x, y, t) \equiv Y|_{\varepsilon=0}$. The theorems formulated in ([1], [2]) pertain to a large time interval $t \sim \frac{1}{\varepsilon}$, but, as simple examples show, they do not carry over directly to an unbounded interval $[t_0, \infty)$. In the present article a theorem is given on averaging on an unbounded interval, connected with additional requirements of asymptotic stability of certain solutions of the degenerate system and of the averaged system of the first approximation.

We introduce the following requirements:

1) Suppose that for $|x| \leq \nu$, $|z| \leq \eta$, $t_0 \leq t < \infty$ (ν and η are certain positive constants) there is defined a Lyapunov function $V(x, z, t)$, continuous jointly in x, y, t, possessing partial derivatives with respect to all variables, with $\partial V/\partial x$ and $\partial V/\partial z$ uniformly bounded. Suppose $V(x, 0, t) \equiv 0$, $V|_{z \neq 0} > 0$, and $V(x, z, t)$ is a positive-definite function and admits an infinitesimal upper bound. Moreover, suppose that the derivative

$$\frac{d}{dt} V(x, z, t)$$

with respect to the system $x = \text{const}$,

$$\dot{z} = Z(x, z, t) \equiv Y_0[x, \varphi(x, t) + z, t] - Y_0[x, \varphi(x, t), t],$$

where $y = \varphi(x, t)$ is some solution of the degenerate system $x = \text{const}$, $\dot{y} = Y_0$, is a negative-definite function (according to the stated conditions, $y = \varphi(x, t)$ is an asymptotically stable solution).

2) Suppose the function $\varphi(x, t)$ is defined for $|x| \leq \nu$, $t \in [t_0, \infty)$, is continuous together with its partial derivatives with respect to x, t, and the derivatives $\partial\varphi/\partial x$ are uniformly bounded.
3) The function $Y(x, y, t, \varepsilon)$ is given for $|x| \leq \nu$, $z \equiv y - \varphi(x, t) \in [-\eta, +\eta]$, $t_0 \leq t < \infty$, $0 \leq \varepsilon \leq \varepsilon_0$ ($\varepsilon_0 > 0$), and is continuous in ε uniformly with respect to x, y, t, ε.
4) The function $Y_0(x, y, t) \equiv Y|_{\varepsilon=0}$ is defined for $|x| \leq \nu$, $z \equiv y - \varphi(x, t) \in [-\eta, +\eta]$, $t_0 \leq t < \infty$, and satisfies a Lipschitz condition in x with a constant independent of x, y, t.
5) There exists an interval $\Delta t(\varepsilon)$: $t_0 \leq t \leq \bar{t}(\varepsilon)$, $0 < \varepsilon \leq \varepsilon_0$, $\bar{t} > t_0$ (the unboundedness of this interval is allowed, i.e., possibly $\bar{t} = \infty$), on which a solution $x(t, \varepsilon), y(t, \varepsilon)$ of the principal system $\dot{x} = \varepsilon X$, $\dot{y} = Y$ is defined, satisfying the prescribed initial conditions $x(t_0, \varepsilon) = x_0$, $y(t_0, \varepsilon) = y_0$, where x_0, y_0, t_0 is a fixed point chosen sufficiently close to the point $x = 0$, $y = \varphi(0, t_0)$, $t = t_0$, and $|x_0| \leq \nu$, $z_0 \equiv y_0 - \varphi(x_0, t_0) \in [-\eta, +\eta]$ (uniqueness of the solution $x(t, \varepsilon), y(t, \varepsilon)$ is not required here).

is assumed). Let $x(t, \varepsilon)$, $y(t, \varepsilon)$, for $t \in \Delta t(\varepsilon)$, be continuously differentiable with respect to t.

6) The function $X(x, y, t, \varepsilon)$ is defined for $|x| \leq \nu$, $z \equiv y - \varphi(x, t) \in [-\eta, +\eta]$, $t_0 \leq t < \infty$, $0 \leq \varepsilon < \varepsilon_0$, and is continuous in ε uniformly with respect to x, y, t, ε.
7) The function $X_1(x, y, t) \equiv X\big|_{\varepsilon=0}$ is continuous and uniformly bounded, together with its first partial derivatives with respect to x and y.
8) For $|x| \leq \nu$, uniformly with respect to x, there exists the mean value—the limit

$$\bar{X}_1(x) \equiv \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} X_1[x, \varphi(x, t), t]\, dt,$$

and the function $\bar{X}_1(x)$ is continuous and uniformly bounded, together with its partial derivatives with respect to x, and for sufficiently large values of $T > 0$ the inequality

$$\left| \int_{t_0}^{t_0+T} [\{X_1 - \bar{X}_1\}]_{y=\varphi(x,t)}\, dt \right| + \left| \int_{t_0}^{t_0+T} \left[\left\{ \frac{\partial X_1}{\partial x} + \frac{\partial X_1}{\partial y} \frac{\partial \varphi}{\partial x} - \frac{\partial \bar{X}_1}{\partial x} \right\}_{y=\varphi(x,t)} \right] dt \right| \leq$$

$$\leq \text{const} < \infty.$$

9) Let $\bar{X}_1(0) = 0$, and let the point $x = 0$ be an asymptotically stable in the sense of Lyapunov equilibrium point of the system $dx/d\tau = \bar{X}_1(x)$ for $0 \leq \tau < \infty$.

Theorem. *Under the indicated conditions, for an arbitrary* $\delta > 0$ *there exist* $\varepsilon_1(\delta) > 0$ *and* $\mu(\delta) > 0$ *such that, for* $0 < \varepsilon \leq \varepsilon_1(\delta)$, $|x_0| \leq \mu(\delta)$, $|y_0 - \varphi(0, t_0)| \leq \mu(\delta)$, *and values* $t \in \Delta t(\varepsilon)$ *(as was said, the interval* $\Delta t(\varepsilon)$ *may also be the unbounded interval* $[t_0, \infty)$*), the inequalities* $|x(t, \varepsilon)| \leq \delta$, $|y(t, \varepsilon) - \varphi(0, t)| \leq \delta$ *hold, where* $x(t, \varepsilon)$, $y(t, \varepsilon)$ *is a solution of the principal system* $\dot{x} = \varepsilon X(x, y, t, \varepsilon)$, $\dot{y} = Y(x, y, t, \varepsilon)$, *satisfying the initial conditions* $x(t_0, \varepsilon) = x_0$, $y(t_0, \varepsilon) = y_0$.

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2. V. M. Volosov, DAN, **137**, No. 5, 1022 (1961).

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* Bibliography on the question under consideration is available in ($^{1-3}$).

Note: Figure translations are in progress. See original paper for figures.

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