ASYMPTOTICS OF THE EIGENVALUES OF A LINEAR INTEGRAL EQUATION WITH A DISCONTINUOUS KERNEL

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Abstract

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MATHEMATICS

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ASYMPTOTICS OF THE EIGENVALUES OF A LINEAR INTEGRAL EQUATION WITH A DISCONTINUOUS KERNEL

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Consider the integral equations

$$y(x) = \lambda \int_a^b K(x,t) y(t)\, dt, \tag{1}$$

whose kernels are bounded and have a finite discontinuity on the straight line $x = t$. As is known ([1]), the eigenvalues of a linear integral equation with a bounded kernel coincide with the set of zeros of a certain entire function, and the exponent of convergence of this set does not exceed two. The extraction of a subset of kernels with a certain singularity makes it possible, analogously to how this was done in ([2,3]), to obtain exact estimates for the eigenvalues.

With respect to the kernel $K(x,t)$ we make the following assumptions:

1. The kernel $K(x,t)$ and its derivatives up to and including the fourth order are continuous and bounded everywhere in $S: a \le x \le b,\ a \le t \le b$, except for the points of the straight line $x = t$, on which they undergo a finite discontinuity:

$$K_x^{(i)}(x+0,x) - K_x^{(i)}(x-0,x) \equiv p_i(x), \qquad p_0(x) \neq 0, \qquad a \le x \le b,$$

$$i = 0,\ 1,\ 2,\ 3,\ 4.$$

2. The integral equation

$$y(x) + \int_a^b R(x,t) y(t)\, dt = 0,$$

where

$$R(x,t) = \left[\frac{1}{p_0(t)} K(x,t) \exp\left(-\int_t^x \frac{p_1(s)}{p_0(s)}\, ds\right)\right]'_x - p_0(x)\delta(x-t),$$

has no solution different from zero.

3. At least one of the following conditions is fulfilled:

$$3'. \quad \Phi(a,b)[\Phi(a,a)+1] \neq 0.$$

$$3''. \quad \left\{\Phi(a,b)\left[\frac{\Phi'_t(a,a)}{p_0(a)} + \Psi(a,a)\right]\right\}^2 + \left\{(\Phi(a,a)+1)\left[\frac{\Phi'_t(a,b)}{p_0(b)} + \Psi(a,b)\right]\right\}^2 \neq 0,$$

where

$$\Phi(x,t) = \frac{K(x,t)}{p_0(t)} \exp\left(-\int_t^x \frac{p_1(s)}{p_0(s)}\, ds\right) -$$

$$- \int_a^b \frac{K(x,s)}{p_0(s)} \exp\left(-\int_s^x \frac{p_1(\xi)}{p_0(\xi)}\, d\xi\right) (E+L)_s^{-1} R(s,t)\, ds,$$

$$\Psi(x,t) = \int_a^b \frac{K(x,s)}{p_0(s)} \exp\left(-\int_s^x \frac{p_1(\xi)}{p_0(\xi)}\, d\xi\right) (E+L)_s^{-1} \left[\left(\frac{R(s,t)}{p_0(s)}\right)'_s - \right.$$

$$\left. - \int_a^b \left(\frac{R(s,u)}{p_0(s)}\right)'_s (E+L)_u^{-1} R(u,t)\, du\right] ds - (E+L)_x^{-1} R(x,t)$$

(here L is the integral operator with kernel $R(x,t)$).

Theorem 1. If $K(x,t)$ satisfies conditions 1, 2, and 3, then the integral equation (1) has an infinite sequence of eigenvalues $\{\lambda_n\}$, which as $n \to \infty$ have the asymptotics

$$\lambda_n = 2\pi i n \left(\int_a^b p_0(s)\, ds\right)^{-1} \left\{1 + \frac{1}{2\pi i n} \ln \Phi(a,b)[\Phi(a,a)+1]^{-1} + O\left(\frac{1}{n^2}\right)\right\}, \tag{2}$$

if condition $3'$ is satisfied, and

$$\lambda_n = 2\pi i n \left(\int_a^b p_0(s)\, ds \right)^{-1} \left\{ 1 + O\left(\frac{\ln n}{n} \right) \right\}, \tag{3}$$

if condition 3″ is satisfied.

If $K(x,t)$ satisfies conditions 1, 2, and some regular boundary condition $\gamma K(a,t) = K(b,t)$, $a \le t \le b$, then $\gamma = \Phi(a,b) : [\Phi(a,a)+1]$ and, consequently, estimate (2) holds for λ_n.

Condition 2 is essential, since its violation may lead to a different asymptotics. Condition 3 is also necessary in a certain sense. If this condition is not satisfied, the integral equation may have no solutions; for example, $y(x) = \lambda \int_0^x y(t)\, dt$.

In order to clarify the significance of condition 3, let us consider kernels satisfying conditions 1, 2, and condition 4:

4. $R(x,t)$ is infinitely differentiable with respect to x and t, and

$$\left| \frac{\partial^{m+n} R(x,t)}{\partial x^m \partial t^n} \right| \le M a^m b^n, \qquad m, n \ge 0,$$

where a, b, and M are some constants.

Theorem 2. The integral equation (1), whose kernel satisfies conditions 1, 2, and 4, either has no solutions for $|\lambda| > \lambda_0$, or its eigenvalues satisfy one of the two possible asymptotic equalities:

$$\lambda_n = 2\pi i n \left(\int_a^b p_0(s)\, ds \right)^{-1} \left[1 + O\left(\frac{1}{n} \right) \right],$$

or

$$\lambda_n = 2\pi i n \left(\int_a^b p_0(s)\, ds \right)^{-1} \left[1 + O \frac{\ln n}{n} \right].$$

For the proof, set $p_0(x) \equiv 1$, $p_1(x) \equiv 0$. The general case is easily reduced to this one by a change of variables and of the unknown function in the integral equation. Then, differentiating both sides of equation (1), we obtain:

$$y'(x) = \lambda y(x) + \lambda \int_a^b R(x,t) y(t)\, dt. \tag{4}$$

An arbitrary solution of the integro-differential equation (4) can be written in the form

$$y(x) = Ce^{\lambda x} + \int_a^b R(x,t)y(t)\,dt + \int_a^b \mathcal{R}(x,t;\lambda)y(t)\,dt,$$

or, denoting by L and $\mathcal{L}$ respectively the integral operators with kernels $R(x,t)$ and $\mathcal{R}(x,t;\lambda)$,

$$(E + L + \mathcal{L})y(x) = Ce^{\lambda x}.$$

Here

$$\mathcal{R}(x,t;\lambda) = -\sum_{n=1}^{\infty} \frac{1}{\lambda^n} R_x^{(n)}(x,t)$$

is a function regular outside the circle $|\lambda| \le a$, tending to zero as $\lambda \to \infty$.

Condition 2 ensures the existence and boundedness of the operator $(E+L)^{-1}$, and by virtue of condition 4 the norm of the operator $\mathcal{L}$ tends to zero as $\lambda \to \infty$. Therefore, for sufficiently large λ, $|\lambda| > \lambda_0$, the operator $(E + L + \mathcal{L})^{-1}$ exists, is bounded, and satisfies the identity

$$(E + L + \mathcal{L})^{-1} = (E + L)^{-1} \sum_{n=0}^{\infty} (-1)^n [\mathcal{L}(E + L)^{-1}]^n.$$

It is not difficult to show that the operator

$$\sum_{n=1}^{\infty} (-1)^n [\mathcal{L}(E + L)^{-1}]^n$$

admits the representation

$$\sum_{n=1}^{\infty} (-1)^n [\mathcal{L}(E + L)^{-1}]^n f(x) = \int_a^b Q(x,t;\lambda) f(t)\,dt,$$

where $f(x)$ is an arbitrary continuous function; $Q(x,t;\lambda)$ is infinitely differentiable with respect to x and t; for any m and n,

$$\frac{\partial^{m+n}}{\partial x^m \partial t^n} Q(x,t;\lambda)$$

is regular in λ outside some circle $|\lambda| \le \lambda_1$, and

$$\left|\frac{\partial^{m+n}Q(x,t;\lambda)}{\partial x^m \partial t^n}\right| \leq M(\lambda)a^m b^m,$$

with $M(\lambda) \to 0$ as $\lambda \to \infty$.

Consequently, the solutions of equation (4) can be written in the form

$$y(x) = C(E+L)^{-1}\left[e^{\lambda x} + \int_a^b Q(x,t;\lambda)e^{\lambda t}\,dt\right]. \tag{5}$$

It is clear that the solutions of the integral equation (1) are contained in formula (5). In order to separate them out, substitute equality (5) into the original integral equation. Since, for an arbitrary solution of equation (4), the difference

$$y(x) - \lambda\int_a^b K(x,t)y(t)\,dt$$

will be some constant, in order that the function defined by formula (5) be a solution of equation (1), it is sufficient that the condition

$$y(a) = \lambda\int_a^b K(a,t)y(t)\,dt \tag{6}$$

be fulfilled.

Since

$$\int_a^b K(a,t)(E+L)^{-1}e^{\lambda t}\,dt = \int_a^b \Phi(a,t)e^{\lambda t}\,dt,$$

then by integration by parts we transform equality (6) into the form

$$\Phi(a,b)e^{\lambda b} - [\Phi(a,a)+1]e^{\lambda a} - \int_a^b [\Phi'_t(a,t) - (E_\lambda^x + L)^{-1}R(a,t)]e^{\lambda t}\,dt +$$

$$+\int_a^b \left\{(E+L)^{-1}Q(a,t;\lambda) - \lambda\int_a^b K(a,s)(E+L)^{-1}Q(s,t;\lambda)\,ds\right\}e^{\lambda t}\,dt = 0. \tag{7}$$

Denoting

$$H(x,t) \equiv \Phi_t'(x,t) - (E+L)^{-1}R(x,t), \quad G(x,t;\lambda) =$$

$$= (E+L)^{-1}Q(x,t;\lambda) - \lambda \int_a^b K(x,s)(E+L)^{-1}Q(s,t;\lambda)\, ds$$

and integrating by parts, we expand the integrals in formula (7) in a series in powers of $1/\lambda$. Then

$$e^{\lambda(b-a)} \left\{ \Phi(a,b) - \sum_{k=0}^{\infty} (-1)^k \frac{H_t^{(k)}(a,b) + G_t^{(k)}(a,b;\lambda)}{\lambda^{k+1}} \right\} -$$

$$- \left\{ \Phi(a,a) + 1 - \sum_{k=0}^{\infty} (-1)^k \frac{H_t^{(k)}(a,a) + G_t^{(k)}(a,a;\lambda)}{\lambda^{k+1}} \right\} = 0 \tag{8}$$

or

$$e^{\lambda(b-a)}\varphi_1(\lambda) - \varphi_2(\lambda) = 0, \tag{9}$$

where $\varphi_1(\lambda)$, $\varphi_2(\lambda)$ are regular outside some circle $|\lambda| \leq \lambda_2$. Therefore either $\varphi_i(\lambda)$ are identically equal to zero, or there exists such a λ_3 that outside the circle $|\lambda| \leq \lambda_3$, $\varphi_i(\lambda) \neq 0$, except, perhaps, for $\lambda = \infty$.

The following cases may occur. a) The functions $\varphi_i(\lambda)$, $i = 1, 2$, have no zeros for $|\lambda| > \lambda_3$. Then $e^{\lambda(b-a)} = \varphi_2(\lambda)/\varphi_1(\lambda)$. It is not difficult to show that the latter equation has a countable set of solutions with the asymptotics indicated in the formulation of Theorem 2. b) One of the functions $\varphi_i(\lambda)$ is identically equal to zero. Equation (9) has no solutions. c) Both functions $\varphi_i(\lambda)$ are identically equal to zero. This case cannot be realized.

If condition 4 is discarded, as was done in Theorem 1, then instead of the infinite series in powers of $1/\lambda$ in formula (8) we obtain only its finite segment, whose length depends on the smoothness of the function $R(x,t)$. In the general case the principal terms of this series will be

$$\Phi(a,b)e^{\lambda(b-a)} - \Phi(a,a) - 1 - \frac{1}{\lambda}\{[\Phi_t'(a,b) + \Psi(a,b)]e^{\lambda(b-a)}$$

$$-[\Phi_t'(a,a) + \Psi(a,a)]\} + \ldots + O\left(\frac{e^{\lambda(b-a)} + 1}{\lambda^n}\right) = 0. \tag{10}$$

Hence conditions $3'$ and $3''$.

The structure of the equation for the eigenvalues (equation (8), if $R(x,t)$ is infinitely differentiable, and (10), if $R(x,t)$ has $2n+1$ derivatives) makes natural the assumption, confirmed by Theorems 1 and 2, that either an integral equation with a kernel satisfying conditions 1 and 2 has no solutions ($|\lambda| > \lambda_0$), or its eigenvalues have the asymptotics indicated in Theorem 2; there can be no other asymptotics.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.