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Abstract

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L. A. SHEMETKOV

ON EMBEDDING THEOREMS AND MAXIMAL SUBGROUPS OF FINITE GROUPS

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§ 1. The main results of the present note are embedding theorems for subgroups of finite groups, connected with not necessarily Hall subgroups (by a Hall subgroup we mean a subgroup whose order is relatively prime to its index).

Theorem 1 generalizes Theorem 15 of Tibilletti ([1]); Theorem 3 generalizes a theorem of H. Wielandt ([2]); Theorems 7-11 generalize the results of the work of S. A. Rusakov ([3]); Theorem 13 generalizes Theorems 2 and 3 of the work of S. A. Chunikhin ([4]); Theorem 14 generalizes Hall' s Theorem D5 ([5]). In § 6 analogues are given of a theorem of H. Wielandt ([2]) connected with the "isomorphic" embedding of subgroups; in § 7 the influence, in a special case, of properties of maximal subgroups on the properties of the group is considered; in § 8 Iwasawa' s theorem on finite J-groups ([6]) is generalized; in § 9 "Π-theorems" corresponding to Theorems 22-24 of the work ([7]) are given.

§ 2. Let Π be some (empty or nonempty) set of primes; $\mathfrak{G}$ a finite group of order $(\mathfrak{G}) = g = mn$, where $m \geqslant 1$ is the greatest Π-divisor ([8]) of the order g; if $\mathfrak{G}$ has at least one subgroup of order m, then by $\mathfrak{G}_\Pi$ we shall denote some subgroup of order m of the group $\mathfrak{G}$; for $m > 1$ put p equal to the greatest prime divisor of the number m, and for $m = 1$ put $p = 1$; $\mathfrak{E}$ is the identity subgroup of the group $\mathfrak{G}$; a Πd-group is a group whose order is divisible by some prime from Π; $h_{\Pi'}$ is the greatest Π'-divisor of the natural number h (Π' is the set of all primes not belonging to Π);

$$h = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}, \quad k \geqslant 1,$$

is the canonical decomposition of the natural number $h > 1$.

Let ρ denote the set of all primes with some ordering introduced on it. A group $\mathfrak{H}$ of order

$$h = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}, \quad k \geqslant 1,$$

will be called strongly ρ-dispersive if the order of the normalizer in $\mathfrak{H}$ of each of its p_i-subgroups ($i = 1, 2, \dots, k$) is divisible by $(\mathfrak{G})_{\Pi_i}$, where Π_i is the set of all those primes which follow p_i in the given ordering. Obviously, a strongly ρ-dispersive group is ρ-dispersive ([9]). We shall regard $\mathfrak{E}$ as a trivially strongly ρ-dispersive group.

The notions of p-speciality, Π-solvability, Π-separability, and strong Π-solvability are taken by us from the work ([10]). For groups of type S, see ([11],[12]). We also use the notion of the indexial of a finite group and the definitions and notation connected with it from the work ([13]).

In what follows, by σ and τ we shall denote such (empty or nonempty) sets of primes that $\sigma \cap \tau$ is empty and $\Pi = \sigma \cup \tau$. The notation $\mathfrak{G}_\sigma$, $\mathfrak{G}_\tau$, $\mathfrak{H}_\sigma$, and so on, is defined analogously to $\mathfrak{G}_\Pi$.

§ 3. Let h be some divisor of the order of the group $\mathfrak{G}$. We introduce for consideration the following properties of finite groups: $E(h)$—in $\mathfrak{G}$ there is at least one subgroup $\mathfrak{H}$ of order h; $C(h)$—$\mathfrak{G}$ has property $E(h)$ and any two subgroups of order h of the group $\mathfrak{G}$ are conjugate in $\mathfrak{G}$; $D(h)$—$\mathfrak{G}$ has property $C(h)$, and every subgroup of order dividing h, of the group ⋯

if $\mathfrak{G}$ is contained in some subgroup of order h of the group $\mathfrak{G}$; $D^s(h)$—$\mathfrak{G}$ has property $D(h)$ and its subgroups of order h are soluble; $D^{ss}(h)$—$\mathfrak{G}$ has property $D(h)$ and its subgroups of order h are supersoluble.

If $h = m$, then instead of $E(h), C(h), D(h), D^s(h)$, and $D^{ss}(h)$ we shall use, respectively, the symbols $E_\Pi, C_\Pi, D_\Pi, D_\Pi^s$, and D_Π^{ss}. If $\mathfrak{A}$ is a subgroup of order a of the group $\mathfrak{G}$, then, along with the symbols $C(a), D^s(a)$, etc., we shall also use the symbols $C(\mathfrak{A}), D^s(\mathfrak{A})$, etc.

We shall say that the strong $D(h)$-theorem holds for the group $\mathfrak{G}$ if $\mathfrak{G}$ has property $D(\mathfrak{L})$ for each of its subgroups $\mathfrak{L}$ whose order divides h. If $h = m$, then we shall speak of the strong D_Π-theorem ([3]).

§ 4. A divisor $h = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $k \geqslant 1$, of the order of the group $\mathfrak{G}$ will be called a quasi-Hall divisor of the order of the group $\mathfrak{G}$ if, for each $i = 1, 2, \dots, k$, the following condition holds: the normalizer $\mathfrak{N}$ of every subgroup $\mathfrak{A}$ of $\mathfrak{G}$ such that $(\mathfrak{A})$ divides $h/p_i^{\alpha_i}$ and $p_i^{\alpha_i}$ divides $(\mathfrak{N})$, has property $C(p_i^{\alpha_i})$. We shall also regard the identity as a quasi-Hall divisor of the order of $\mathfrak{G}$.

A subgroup $\mathfrak{H}$ of the group $\mathfrak{G}$ will be called a quasi-Hall subgroup of the group $\mathfrak{G}$ if $(\mathfrak{H})$ is a quasi-Hall divisor of the order of $\mathfrak{G}$. It is easy to see that every Hall subgroup of the group $\mathfrak{G}$ is also its quasi-Hall subgroup.

Theorem 1. Let $\mathfrak{G}$ have a quasi-Hall ρ-dispersive subgroup $\mathfrak{H}$ of order h, and let $\mathfrak{L}$ be a ρ-dispersive subgroup of order l, dividing h, of the group $\mathfrak{G}$, with $(l, h/l) = 1$. Then there is an element $G \in \mathfrak{G}$ such that $\mathfrak{L}^G \supseteq \mathfrak{H}$.

Theorem 2. Let $\mathfrak{G}$ have a quasi-Hall strongly ρ-dispersive subgroup $\mathfrak{H}$. Then for every ρ-dispersive subgroup $\mathfrak{L}$ of order dividing $(\mathfrak{H})$ of the group $\mathfrak{G}$, there is an element $G \in \mathfrak{G}$ such that $\mathfrak{L}^G \subseteq \mathfrak{H}$.

Theorem 3. Let $\mathfrak{G}$ have a quasi-Hall subgroup $\mathfrak{H} = \mathfrak{H}_\sigma \times \mathfrak{H}_\tau$, where $\mathfrak{H}_\sigma$ is special, and suppose $\mathfrak{G}$ has property $D(\mathfrak{H}_\tau)$. Then $\mathfrak{G}$ has property $D(\mathfrak{H})$.

Theorem 4. Let $\mathfrak{G}$ have a quasi-Hall subgroup $\mathfrak{H} = \mathfrak{H}_\sigma \times \mathfrak{H}_\tau$, where $\mathfrak{H}_\sigma$ is strongly ρ-dispersive, and suppose $\mathfrak{G}$ has properties $D(\mathfrak{H}_\sigma)$ and $D(\mathfrak{H}_\tau)$. Then

$\mathfrak{G}$ has property $D(\mathfrak{H})$.

Theorem 5. Let $\mathfrak{G}$ have a subgroup $\mathfrak{G}_\Pi = \mathfrak{G}_\sigma \times \mathfrak{G}_\tau$, where $\mathfrak{G}_\sigma$ is strongly ρ-dispersive, and suppose $\mathfrak{G}$ has properties D_σ and D_τ. Then $\mathfrak{G}$ has property D_Π.

Theorem 6. Let $\mathfrak{G}$ have a subgroup $\mathfrak{H} = \mathfrak{H}_\sigma \times \mathfrak{G}_\tau$, where $\mathfrak{H}_\sigma$ is ρ-dispersive. Suppose that $\mathfrak{G}$ and all its subgroups containing $\mathfrak{H}_\sigma$ have property $D(\mathfrak{H}_\sigma)$, and suppose that $\mathfrak{G}$ has property D_τ. Then $\mathfrak{G}$ has property $D(\mathfrak{H})$.

Theorem 7. Let $h = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $k \geqslant 1$, be a quasi-Hall divisor of the order of $\mathfrak{G}$, and suppose that, for each $i = 1, 2, \dots, k$, $\mathfrak{G}$ has a cyclic subgroup of order $p_i^{\alpha_i}$. Then the strong $D(h)$-theorem holds for $\mathfrak{G}$.

Theorem 8. Let $\mathfrak{G}$ have a quasi-Hall subgroup $\mathfrak{H} = \mathfrak{H}_\sigma \times \mathfrak{H}_\tau$, where all Sylow subgroups of $\mathfrak{H}_\sigma$ are cyclic, and suppose $\mathfrak{G}$ has property $D(\mathfrak{H}_\tau)$. Then $\mathfrak{G}$ has property $D(\mathfrak{H})$.

Theorem 9. Suppose that all Sylow subgroups of the group $\mathfrak{G}$ corresponding to primes in σ are cyclic. Let $\mathfrak{G}$ have a subgroup $\mathfrak{H} = \mathfrak{H}_\sigma \times \mathfrak{G}_\tau$, and suppose $\mathfrak{G}$ has property D_τ. Then $\mathfrak{G}$ has property $D(\mathfrak{H})$.

Theorem 10. Let $\mathfrak{G}$ have a quasi-Hall subgroup $\mathfrak{H} = \mathfrak{H}_\sigma \times \mathfrak{H}_\tau$, where all Sylow subgroups of $\mathfrak{H}_\sigma$ are cyclic, and suppose that the strong $D(\mathfrak{H}_\tau)$-theorem holds for $\mathfrak{G}$. Then the strong $D(\mathfrak{H})$-theorem holds for $\mathfrak{G}$.

Theorem 11. Let all Sylow subgroups of the group $\mathfrak{G}$ corresponding to the prime numbers in σ be cyclic. Let $\mathfrak{G}$ have a subgroup $\mathfrak{H} = \mathfrak{H}_\sigma \times \mathfrak{G}_\tau$, and let the strong D_τ-theorem hold for $\mathfrak{G}$. Then the strong $D(\mathfrak{H})$-theorem holds for $\mathfrak{G}$.

§ 5. A divisor $h = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $k \geq 1$, of the order of a group $\mathfrak{G}$ such that $\mathfrak{G}$ has the property $C(p_i^{\alpha_i})$ for each $i = 1, 2, \dots, k$, and also 1, will be called a quasi-Sylow divisor of the order of the group $\mathfrak{G}$.

A divisor h of the order of the group $\mathfrak{G}$ such that (h, l) is a quasi-Sylow divisor of the order l of an arbitrary characteristic subgroup Ω of the group $\mathfrak{G}$ (in particular, $\Omega = \mathfrak{G}$) will be called a strong quasi-Sylow divisor of the order of the group $\mathfrak{G}$.

We shall call the indexed set $(h)_{R,f}$ of the group $\mathfrak{G}$ Hall if: 1) $(h)_{R,f}$ is a refined indexed set of the group $\mathfrak{G}$; 2) $\mathfrak{G}_{\beta-1}/\mathfrak{G}_\beta$ has the property $D^s(f_\beta)$; 3) $\mathfrak{F}_i/\mathfrak{G}_i$ is a special Hall subgroup of the group $\mathfrak{G}_{i-1}/\mathfrak{G}_i$, $i = \beta+1, \beta+2, \dots, \nu$.

Theorem 12. Let h be such a quasi-Sylow divisor of the order of a Π-solvable group $\mathfrak{G}$ that $h_{\Pi'}$ is equal either to 1 or to n. Then $\mathfrak{G}$ has the property $D(h)$.

Theorem 13. Let h be such a strong quasi-Sylow divisor of the order of a Π-solvable group $\mathfrak{G}$ that $h_{\Pi'}$ is equal either to 1 or to n. Then $\mathfrak{G}$ has the property $D(h)$.

Theorem 14. Let $(h)_{R,f}$ be a Hall indexed set of the group $\mathfrak{G}$, and let h be a quasi-Sylow divisor of the order of $\mathfrak{G}$. Then $\mathfrak{G}$ has the property $D^s(h)$.

§ 6. We shall say that a group $\mathfrak{G}$ has the property $I(h)$ if $\mathfrak{G}$ has at least one subgroup $\mathfrak{H}$ of order h, and for any subgroup Ω of the group $\mathfrak{G}$ whose order divides h, there is a subgroup Ω^* of $\mathfrak{H}$ such that Ω and Ω^* are isomorphic. We shall say that the strong $I(h)$-theorem holds for $\mathfrak{G}$ if $\mathfrak{G}$ has the property $I(l)$ for each of its subgroups Ω whose order l divides h. If $h = m$, then instead of $I(h)$ we shall use the symbol I_Π. The notations I_σ and I_τ are introduced analogously. If $\mathfrak{A}$ is a subgroup of order a of the group $\mathfrak{G}$, then along with $I(a)$ we shall also use the symbol $I(\mathfrak{A})$.

Theorem 15. Let $\mathfrak{G}$ have a quasi-Hall subgroup $\mathfrak{H} = \mathfrak{H}_\sigma \times \mathfrak{H}_\tau$, where $\mathfrak{H}_\tau$ is ρ-dispersive, and let $\mathfrak{G}$ have the properties $I(\mathfrak{H}_\sigma)$ and $I(\mathfrak{H}_\tau)$. Then $\mathfrak{G}$ has the property $I(\mathfrak{H})$.

Theorem 16. Let $\mathfrak{G}$ have a quasi-Hall subgroup $\mathfrak{H} = \mathfrak{H}_\sigma \times \mathfrak{H}_\tau$, where $\mathfrak{H}_\sigma$ is ρ-dispersive, and let the strong $I(\mathfrak{H}_\sigma)$-theorem and the strong $I(\mathfrak{H}_\tau)$-theorem hold for $\mathfrak{G}$. Then the strong $I(\mathfrak{H})$-theorem holds for $\mathfrak{G}$.

§ 7. If in the group $\mathfrak{G}$ the set of all maximal subgroups with core $\mathfrak{C}$ ([9]) is nonempty, then this set is divided into classes by the isomorphy relation ([14]). The classes obtained will be called classes of isomorphic maximal subgroups with core $\mathfrak{C}$. We shall say that $\mathfrak{G}$ is a group with one class of isomorphic maximal subgroups with core $\mathfrak{C}$ if $\mathfrak{G}$ has only one class of isomorphic maximal subgroups with core $\mathfrak{C}$.

Theorem 17. Let $\mathfrak{G}$ be a group with one class of isomorphic maximal subgroups with core $\mathfrak{C}$. If among the isomorphic subgroups of the given class there is at least one subgroup possessing the property E_Π^n ([5]), then $\mathfrak{G}$ has the property D_Π^s.

Theorem 18. Let $\mathfrak{G}$ be a group with one class of isomorphic maximal subgroups with core $\mathfrak{C}$. If among the isomorphic subgroups of the given class there is at least one Π-solvable or Π-separable subgroup, then $\mathfrak{G}$ is respectively Π-solvable or Π-separable.

§ 8. If $\mathfrak{G} \neq \mathfrak{C}$, then a series of subgroups

$$\mathfrak{G} = \mathfrak{H}_0 \supset \mathfrak{H}_1 \supset \cdots \supset \mathfrak{H}_t = \mathfrak{C}, \qquad t \geq 1,$$

will be called a maximal series of the group $\mathfrak{G}$, if each term of this series

$\mathfrak{H}_i$ $(i = 1, 2, \dots, t)$ is a maximal subgroup of the preceding member of the series $\mathfrak{H}_{i-1}$. If the group $\mathfrak{G} = \mathfrak{C}$, then its only maximal series will be considered to be $\mathfrak{C}, \mathfrak{C}$. If in every maximal series of the group $\mathfrak{G}$ the number of indices, all prime divisors of which belong to Π, is the same, then $\mathfrak{G}$ will be called a ΠJ-group.

Theorem 19. If $\mathfrak{G}$ is strongly Π-solvable, then it is a ΠJ-group.

Theorem 20. If $\mathfrak{G}$ is a ΠJ-group and $(p!, n) = 1$, then $\mathfrak{G}$ is strongly Π-solvable.

Theorem 21. If a ΠJ-group $\mathfrak{G}$ has property E_Π, then it also has property D_Π^{ss}.

§ 9. In the present paragraph $\mathfrak{G}$ will denote a group for which $m > 1$ and $n > 1$. A subgroup $\mathfrak{H}$ of the group $\mathfrak{G}$ will be called an r-th Π-maximal subgroup of the group $\mathfrak{G}$, if there exists a maximal series of the group $\mathfrak{G}$

$$\mathfrak{G} = \mathfrak{H}_0 \supset \mathfrak{H}_1 \supset \cdots \supset \mathfrak{H}_r = \mathfrak{H} \supset \cdots \supset \mathfrak{H}_t = \mathfrak{E},$$

where $\mathfrak{H}_i$ is a Πd-subgroup for every $i = 1, 2, \ldots, r-1$.

Theorem 22. If all proper subgroups of the group $\mathfrak{G}$ are strongly Π-solvable and $(p!, n) = 1$, then $\mathfrak{G}$ is either strongly Π-solvable or a p-special group of type S.

Theorem 23. If in a nonspecial group $\mathfrak{G}$ all its second Π-maximal subgroups are invariant, then $\mathfrak{G}$ is either a strongly p-solvable group of type S, or the direct product of a group of order $m = p$ by a group of type S.

Theorem 24. If in a group $\mathfrak{G}$ all its third Π-maximal subgroups are invariant and $(p!, n) = 1$, then $\mathfrak{G}$ is either strongly Π-solvable or a p-special group of type S.

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Gomel Branch
of the Institute of Mathematics and Computational Technology
of the Academy of Sciences of the BSSR

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Note: Figure translations are in progress. See original paper for figures.