A. DZHURAEV

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Abstract

Full Text

A. DZHURAEV

THE GENERAL LINEAR BOUNDARY-VALUE PROBLEM FOR THE EQUATION

$$\Delta u + \lambda c(x, y)u = 0$$

(Presented by Academician I. N. Vekua, 9 X 1961)

1. Let D denote a bounded domain of class ${}^{(1)}C_\alpha$, $0 < \alpha < 1$, in the plane of the variables x, y; let Γ be its boundary. Denote by L_n an arbitrary linear differential operator of order n with variable real coefficients, Hölder-continuous on Γ,

$$L_n \equiv \sum_{k=1}^{n} \sum_{l=0}^{k} a^{k-l,l}(\xi, \eta) \frac{\partial^k}{\partial \xi^{k-l} \partial \eta^l}, \qquad \xi, \eta \in \Gamma,$$

and by $\overset{0}{L}_n$ the principal part of the operator L_n. Assuming $c(x, y)$ to be a real-valued function of class ${}^{(1)}C_\alpha^{n-1}(D + \Gamma)$, in the present note we shall study the following boundary-value problem:

Problem C_λ. Find in D real solutions of the equation

$$\Delta u + \lambda c(x, y)u = 0, \tag{1}$$

continuous together with their derivatives up to order n inclusive in the closed domain $\overline{D} = D + \Gamma$, and satisfying on the boundary Γ the general linear condition

$$\left(\overset{0}{L}_n + \lambda L_{n-1} \right) u(\xi, \eta) = h(\xi, \eta), \tag{2}$$

where $h(\xi, \eta)$ is a prescribed real-valued function on Γ.

Problem C_1 was first formulated for a second-order equation of elliptic type, reduced to canonical form, and studied in [(2)] under the assumption of analyticity of the coefficients of the equation, by methods of the theory of one-dimensional singular equations. Dispensing with the analyticity requirement, we shall study problem C_λ, reducing it to a certain Riemann–Hilbert boundary-value problem for a system of elliptic type. This method was first applied in [(3)] in the study of problems with oblique derivative for generalized analytic functions*. In what follows, as in [2], we shall assume that problem C_λ is normal, i.e., everywhere on the contour Γ the condition

$$a^*(t) = \sum_{l=0}^{n} i^l a^{n-l,l}(\xi, \eta) \neq 0, \qquad t = \xi + i\eta \in \Gamma. \tag{3}$$

is satisfied.

Taking into account the continuity, together with the derivatives up to Γ, of the sought function $u(x, y)$, equation (1) and the boundary condition (2) can be brought to the form

$$\partial^2 u/\partial z\, \partial \overline{z} + \lambda c_0(x, y) u = 0, \tag{4}$$

$$2\partial/\partial \overline{z} = \partial/\partial x + i\partial/\partial y, \qquad 2\partial/\partial z = \partial/\partial x - i\partial/\partial y;$$

$$\operatorname{Re}\left[a^*(t)\frac{\partial^n u}{\partial t^n} + \sum_{k=0}^{n-1} a_k(t, \lambda)\frac{\partial^k u}{\partial t^k}\right] = h_0(t), \qquad c_0 = \frac{1}{2}c, \quad h_0 = \frac{1}{2}h, \tag{5}$$

where $a_k(t, \lambda)$ are completely determined functions satisfying the condition

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* In the dissertation of Teng En Cher, problem C_1 was studied for $n = 1$ for a multiply connected domain [7].

Hölder in t and analytic in λ (polynomials in λ), such that $a_k(t, 0) \equiv 0$, $k = 0, 1, 2, \dots, n-1$. These functions are uniquely determined by the coefficients of the operator L_n and by the boundary values of the function $c(x, y)$ and its derivatives up to order $n-2$ inclusive. To study problem C_λ, consider the following auxiliary problem.

Problem P_λ. Find, continuous in the closed domain $\overline{D}$, a complex-valued vector function

$$V(z) = [v_1, v_2, \dots, v_{2n}],$$

satisfying in D the equation

$$\partial V/\partial \overline{z} + A^\lambda V + B^\lambda \overline{V} = 0 \tag{6}$$

and the boundary condition on Γ

$$\operatorname{Re}[G_\lambda(t) V(t)] = \mathscr{H}_0(t), \qquad \mathscr{H}_0 = [0, h_0, 0, \dots, 0], \tag{7}$$

where $A^\lambda(z) = \|A_{kl}\|$, $B^\lambda(z) = \|B_{kl}\|$, $G_\lambda(t) = \|G_{kl}\|$ are square $(2n \times 2n)$ matrices such that:

$$A_{kl} = \lambda C_{k-2}^{l-1} \partial^{k-l-1} c_0(x, y)/\partial z^{k-l-1}, \quad k = 3, 4, \dots, n+1; \quad l = 2, 3, \dots, n;$$

$$B_{1,1} = 0, \quad B_{k,1} = \lambda \partial^{k-2} c_0(x,y)/\partial z^{k-2}, \quad k = 2, 3, \dots, n-1;$$
$$B_{1,2} = B_{n+2,n+1} = B_{n+3,n} = \dots = B_{2n,3} = -1,$$
$$G_{1,1} = G_{4,n} = G_{4,n+2} = G_{6,n-1} = G_{6,n+3} = \dots$$
$$\dots = G_{2n,2} = G_{2n,2n} = i, \quad G_{2,k} = a_{k-1}(t,\lambda), \quad G_{2,n+1} = a^*(t), \quad k = 1, 2, 3, \dots, n;$$
$$G_{3,n+1} = G_{5,n+2} = \dots = G_{2n-1,2n} = 1, \quad G_{2k-1,n-k+2} = -1, \quad k = 2, 3, \dots, n,$$

and all their remaining elements are equal to zero. Here C_{k-2}^{l-1} denotes the binomial coefficient from $k-2$ elements taken $l-1$ at a time.

Theorem 1. If $u(x,y)$ is a solution of problem C_λ, then the vector function $V(z)$ with components

$$v_1 = u, \quad v_k = \partial^{k-1} u/\partial z^{k-1}, \quad k = 2, 3, \dots, n+1;$$
$$v_k = \partial^{2n-k+1} u/\partial z^{2n-k+1}, \quad k = n+2, \dots, 2n,$$

will be a solution of problem P_λ. Conversely, if the vector function V is a solution of problem P_λ, and if the homogeneous Dirichlet problem for equation (1) has no nonzero solutions, then the first component v_1 of the vector function $V(z)$ will be a solution of problem C_λ.

Since

$$\det G_\lambda(t) = (2i)^n i a^*(t) \neq 0,$$

problem P_λ is normal. Therefore, applying to it the results of work ([4]), and then comparing linearly independent solutions of the problems C_λ and P_λ, we obtain the following result for an $(m+1)$-connected domain D:

Theorem 2. For the solvability of problem C_λ it is necessary and sufficient that the conditions

$$\int_\Gamma h(t) v_j(t)\, dt = 0, \quad j = 1, 2, \dots, \hat{l},$$

hold, where $v_j(t)$ are certain linearly independent functions. The number of solvability conditions $\hat{l}$ of problem C_λ is equal to

$$\hat{l} = l_C - 2[\varkappa + n(m-1)],$$

where

$$\varkappa = \frac{1}{2\pi}\{\arg a^*(t)\}_\Gamma$$

is the index of problem C_λ; l_C is the number of linearly independent solutions of the homogeneous problem C_λ^0 $(h = 0)$. The number l_C is finite and bounded below:

$$l_C \geq \max(0, 2[\varkappa - n(m-1)]) - q,$$

where q is a nonnegative integer $\leq l_D$, and l_D is the number of linearly independent solutions of the homogeneous Dirichlet problem for equation (1). For $n = 1$,

$$l_C = \max(0, 2[\varkappa - n(m-1)]) - q, \quad \varkappa < 0.$$

2. Let D be simply connected ($m = 0$). In this case, without loss of generality, one may assume that D is the unit disk. Then the boundary matrix $G_\lambda(t)$ can be represented in the form

$$G_\lambda = E_\Omega \cdot G_\chi \cdot G_\varphi,$$

where $E_\Omega = \|E_{kk}\|$, $G_\chi = \|G_{kk}\|$, $k = 1, 2, \ldots, 2n$, are diagonal matrices, moreover such that

$$E_{kk} = G_{kk} = 1, \quad k \neq 2; \qquad E_{22} = \exp(-\Omega), \quad G_{22} = t^{-\varkappa};$$

$$\Omega = \operatorname{Im} \varphi; \quad \varphi(z) = \frac{1}{2\pi i} \int_\Gamma [\arg a^*(t) + \varkappa \arg t] \frac{t+z}{t-z} \frac{dt}{t};$$

$$\widehat{G}_\varphi = \|\widehat{G}_{kl}\|$$

is a square $(2n \times 2n)$ matrix all of whose elements are zero except

$$\widehat{G}_{2,k} = a_{k-1}(t, \lambda)/a^*(t), \quad k = 1, 2, \ldots, n;$$

$$\widehat{G}_{2k-1,n-k+2} = -1, \quad k = 2, 3, \ldots, n; \quad \widehat{G}_{3,n+1} = \widehat{G}_{5,n+2} = \ldots = \widehat{G}_{2n-1,2n} = 1;$$

$$\widehat{G}_{2,n+1} = e^{i\varphi(z)}; \quad \widehat{G}_{11} = \widehat{G}_{4,n} = \widehat{G}_{4,n+2} = \widehat{G}_{6,n-1} = \widehat{G}_{6,n+3} = \ldots$$

$$= \ldots = \widehat{G}_{2n,2} = \widehat{G}_{2n,2n} = i.$$

Let us now suppose that the coefficients of the operator L_n are continuously differentiable. Then, since $\det \widehat{G}_\varphi(t) = (2i)^n i e^{i\varphi(t)} \neq 0$, and φ is a single-valued function, we have $\{\arg \det \widehat{G}_\varphi(t)\}_\Gamma = 0$, and the matrix can be extended into D in such a way that, if $\widehat{G}_\varphi(z)$ is its extension, then $\det \widehat{G}_\varphi(z) \neq 0$ and $\widehat{G}_\varphi(z)$ is continuously differentiable. Making now the transformation $\widehat{G}_\varphi(z)V(z) = W(z)$, we bring equation (6) and condition (7) to the form

$$\partial W/\partial \bar{z} + P^\lambda W + Q^\lambda \overline{W} = 0, \tag{8}$$

$$\operatorname{Re}[G_\chi(t)W(t)] = \mathcal{H}(t), \qquad \mathcal{H} = E_\Omega^{-1} \mathcal{H}_0, \tag{9}$$

in which the matrices P^λ, Q^λ depend analytically on the parameter λ in such a way that $P^0(z) \equiv 0$, while the matrix $Q^0(z)$ has the form $Q^0(z) = \|Q_{kl}\|$, where

$$Q_{32} = -e^{i\overline{\varphi(z)}}, \qquad Q_{42} = -ie^{i\overline{\varphi(z)}},$$

$$Q_{2\bar{k}-1,\,2\bar{k}-3} = 1/2; \qquad Q_{2\bar{k}-1,\,2\bar{k}-2} = 1/2i, \qquad \bar{k} = 3, 4, \ldots, n;$$

$$Q_{2\tilde{k},\,2\tilde{k}-3} = i/2, \qquad Q_{2\tilde{k},\,2\tilde{k}-2} = 1/2, \qquad \tilde{k} = 4, 5, \dots, n; \tag{10}$$

$$Q_{k,l} \equiv 0, \qquad k \neq 2\bar{k}-1,\, 2\tilde{k}, \quad l \neq 2\bar{k}-3,\, 2\bar{k}-2,\, 2\tilde{k}-3,\, 2\tilde{k}-2.$$

3. Let $\chi \geq 0$. Then the solution of equation (8) can be represented in the form

$$W(z) = T_\lambda W + \Phi(z), \tag{11}$$

where T_λ is a completely continuous operator in the space of complex-valued vectors continuous in D, which has the form

$$T_\lambda W = \frac{1}{\pi} \iint_D \left\{ \frac{\chi_\lambda(t)}{t-z} + z g_\chi(z) \frac{\overline{\chi_\lambda(t)}}{1-\bar{t}z} \right\} dD_t, \qquad \chi_\lambda \equiv P^\lambda W + Q^\lambda \overline{W}, \tag{12}$$

and $\Phi(z)$ is an arbitrary vector-function holomorphic in D and continuous in $\overline{D}$, where $g_\chi(z) = \|g_{kk}(z)\|$, $k = 1, 2, \dots, 2n$, is a diagonal matrix such that $g_{22}(z) = z^{2\chi}$, $g_{kk} = 1$, $k \neq 2$. Since, obviously, $g_\chi(t) = G_\chi^{-1}(t)\overline{G_\chi(t)}$, it is not difficult to verify that $\mathrm{Re}[G_\chi(t) T_\lambda W] = 0$. Therefore, substituting (11) into (9), we see that equation (11) is completely equivalent to problem P_λ, provided the holomorphic vector-function $\Phi(z) = [\Phi_1, \Phi_2, \dots, \Phi_{2n}]$ has the form

$$\Phi_k(z) = i\gamma_k, \qquad k = 1, 3, 4, \dots, 2n, \qquad h_1 = e^{\Omega} h_0,$$

$$\Phi_2(z) = \frac{1}{2\pi i} \int_\Gamma h_1(t) \frac{t+z}{t-z} \frac{dt}{t} + i\alpha_0 z^\chi + \tag{13}$$

$$+ \sum_{l=0}^{\chi-1} \{\alpha_l (z^l - z^{2\chi-l}) + i\beta_l (z^l + z^{2\chi-l})\}.$$

Let $\dot{M}$ denote the set of real numbers consisting of those λ for which the homogeneous equation $W - T_\lambda W = 0$ has nonzero solutions. Then, applying I. Tamarkin's theorem ([5]) to equation (11) and taking (10) into account, we arrive at the following result:

Theorem 3. *If the index of the problem C_λ is nonnegative, then the problem C_λ is always solvable, except, possibly, for a discrete set $\dot{M}$ of values of λ, not containing the point $\lambda = 0$. In particular, the problem C_λ is always solvable for sufficiently small λ. If $\lambda \notin \dot{M}$, then the homogeneous problem C_λ^0 ($h \equiv 0$) has exactly $l_C = 2(\chi + n) - q$ linearly independent solutions. If, however, $\lambda \in \dot{M}$,* *then, for the solvability of the problem C_λ, it is necessary and sufficient that $r - s$ additional conditions be fulfilled, where r is the rank of the eigenvalue λ, and s is a number satisfying the inequality $0 \le s \le \min[r,\, 2(\chi + n) - q]$.*

4. Suppose now that the index $\varkappa$ is negative: $\varkappa = -\varkappa_1$, $\varkappa_1 > 0$. Then the boundary condition (9) can be written in the form

$$\{\mathrm{Re}[\overline{G_{\varkappa_1}(t)}\, W(t)]\} = \mathcal{H}(t), \tag{14}$$

and the solution of equation (8) can be represented in the form

$$W(z) = \frac{1}{2\pi i} \int_\Gamma \frac{W(t)}{t - z}\, dt + \frac{1}{\pi} \iint_D \frac{\chi_\lambda(t)}{t - z}\, dD_t. \tag{15}$$

Transforming the first term on the right-hand side with the use of (14) and Green' s formula, we obtain the following Fredholm integral equation, equivalent to the problem P_λ for $\varkappa < 0$:

$$W - K_\lambda W = \frac{1}{\pi i} \int_\Gamma \frac{G_\varkappa(t)\mathcal{H}(t)}{t - z}\, dt - g_\varkappa(0)\overline{W(0)}, \tag{16}$$

where

$$K_\lambda W = \frac{1}{\pi} \iint_D \left\{ \frac{\chi_\lambda(z)}{t - z} - \frac{g_\varkappa(t)}{t}\, \frac{\overline{\chi_\lambda(t)}}{1 - tz} \right\} dD_t.$$

Investigating equation (16), we arrive at the following theorem:

Theorem 4. Let $\varkappa < 0$. Then the problem C_λ is solvable for all λ except, possibly, for a discrete set M_1 not containing the zero point and, in particular, it is solvable for sufficiently small λ. If $\lambda \notin M_1$, then the homogeneous problem C_λ^0 has no more than $4n - 2 - q$ linearly independent solutions, while the non-homogeneous problem has exactly one solution provided it is normalized by the conditions $\partial^k u / \partial z^k |_{z=0} = 0$, $k = 0, 1, \dots, n$.

If, however, $\lambda \in M_1$, then for solvability of the problem C_λ it is necessary to impose $r_1 - s_1$ additional conditions, where r_1 is the rank of the eigenvalue λ, and s_1 is a number determined by the inequality $0 \le s_1 \le \min(0,\, 4n - 2 - q)$.

5. Let $\hat{C}(x, y) = \|\hat{c}_{kl}\|$ be a square matrix of order N, whose elements are real functions of class $C_\alpha^{n-1}(\overline{D})$, $0 < \alpha < 1$, and let $a^{p,q}(\xi, \eta) = \|a_{kl}^{p,q}\|$ be square matrices of order N, given on Γ, whose elements are real functions continuous in the sense of Hölder. Consider the following problem:

Problem $\widehat{C}_\lambda$. It is required to determine a real vector-function $U = [u_1, u_2, \dots, u_N]$, continuous in $\overline{D}$ together with its derivatives up to order n inclusive, satisfying in D the equation $\Delta U + \lambda \widehat{C}(x, y)U = 0$ and on Γ the condition

$$\left(\sum_{j=0}^{n} a^{n-j,j}(\xi,\eta)\frac{\partial^n}{\partial\xi^{n-j}\partial\eta^j} + \lambda\sum_{j=1}^{n-1}\sum_{\mu=0}^{j} a^{j-\mu,\mu}(\xi,\eta)\frac{\partial^j}{\partial\xi^{j-\mu}\partial\eta^\mu}\right)U = H(\xi,\eta),$$

where $H = [H_1, \dots, H_N]$ is a given real vector-function on Γ, Hölder-continuous.

Let

$$\hat{\mathbf{a}}^*(t) = \sum_{j=0}^{n} i^j a^{n-j,j}(\xi,\eta), \qquad \hat{\varkappa} = \frac{1}{2\pi}\{\arg\det\hat{\mathbf{a}}^*(t)\}_\Gamma.$$

Then, if $\det\hat{\mathbf{a}}^*(t) \neq 0$ everywhere on Γ, Theorems 2, 3, and 4 are valid for the problem $\widehat{C}_\lambda$, provided that in them n is replaced by $n \cdot N$, and $\varkappa$ by $\hat{\varkappa}$; moreover, in the case of Theorems 3 and 4, continuous differentiability is required of the boundary matrices $a^{p,q}$.

Institute of Hydrodynamics
Siberian Branch of the Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.