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Abstract

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ON THE SOLUTION OF THE BASIC BOUNDARY-VALUE PROBLEM FOR CERTAIN LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF EVEN ORDER

(Presented by Academician S. L. Sobolev on 28 V 1962)

In the article ([2]), starting from the simplest differential operator with partial derivatives $\delta u = \partial u/\partial x + \partial u/\partial y$, the concept of a weakly elliptic operator of arbitrary order was introduced. However, the concept of a weakly elliptic operator can be considerably broadened if, as the initial operator, one takes the operator $\delta_{\alpha,\beta} u = \alpha\, \partial u/\partial x + \beta\, \partial u/\partial y$, where α and β are certain nonzero constants. Here we shall restrict ourselves to weakly elliptic operators of even order for functions of two independent real variables.

1. We shall call the operator

$$Au = \sum_{i,k=0}^{l} (-1)^l \frac{\partial^l}{\partial x^{\,l-i} \partial y^{\,i}} \left(a_{ik}(x,y) \frac{\partial^l u}{\partial x^{\,l-k} \partial y^{\,k}} \right), \qquad l \geqslant 1, \tag{1}$$

weakly elliptic in the closed domain $\overline{\Omega}$, if there exists a positive constant μ such that, for given nonzero constants $a_0, \alpha_1, \dots, \alpha_l$, for any numbers $\xi_0, \xi_1, \dots, \xi_l$ and any points $(x,y) \in \overline{\Omega}$, the inequality

$$\sum_{i,k=0}^{l} a_{ik} \xi_i \xi_k \geqslant \mu \left(\sum_{k=0}^{l} C_l^{(k)} \alpha_k \xi_k \right)^2$$

holds.

Every elliptic operator of the form (1) is at the same time a weakly elliptic operator in the indicated sense. Let us give two special types of weakly elliptic operators of the second order which are not elliptic operators in the closed domain $\overline{\Omega}$.

1) Every parabolic operator in $\overline{\Omega}$

$$Au = -\left(a \frac{\partial^2 u}{\partial x^2} + 2b \frac{\partial^2 u}{\partial x\, \partial y} + c \frac{\partial^2 u}{\partial y^2} \right)$$

with constant coefficients a, b, c, where $a > 0$, is at the same time a weakly elliptic operator in $\overline{\Omega}$. Indeed, since $ac = b^2$, we have

$$a\xi_0^2 + 2b\xi_0\xi_1 + c\xi_1^2 = \mu(\alpha_0\xi_0 + \alpha_1\xi_1)^2, \qquad \mu = \frac{1}{a}, \quad \alpha_0 = a, \quad \alpha_1 = b.$$

2) The operator

$$Au = -\frac{\partial}{\partial x}\left[(a + a_{11}(x,y))\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}\right] - \frac{\partial}{\partial y}\left[b\frac{\partial u}{\partial x} + (c + a_{22}(x,y))\frac{\partial u}{\partial y}\right],$$

where $a_{11}(x,y)$ and $a_{22}(x,y)$ are nonnegative differentiable functions in $\overline{\Omega}$, and the numbers a, b, c $(a > 0)$ are related by $ac = b^2$, will also be weakly elliptic in $\overline{\Omega}$. From the classical point of view this operator is elliptic throughout the closed domain $\overline{\Omega}$, except for a finite

number of points whose coordinates satisfy the system of two equations $a_{11}(x,y) = 0$, $a_{22}(x,y) = 0$; at the indicated points it will be parabolic. If, however, one of the functions $a_{11}(x,y)$ and $a_{22}(x,y)$ is identically zero or identically coincides with the other, then in the domain $\overline{\Omega}$ there may be lines of parabolicity of the operator under consideration.

If the finite domain Ω of the (xy)-plane is bounded by a piecewise smooth curve S with a center of symmetry, then for any function $u(x,y)$, continuously differentiable in the closed domain $\overline{\Omega} = \Omega + S$ and vanishing on the curve S, the inequality

$$\iint_\Omega (\delta_{\alpha,\beta} u)^2 \, d\Omega \geq \gamma^2 \iint_\Omega u^2 \, d\Omega, \qquad \gamma = \text{const} > 0. \tag{2}$$

holds.

To obtain this inequality, it is necessary to take into account that $\delta_{\alpha,\beta} u = \sqrt{\alpha^2 + \beta^2}\, du/d\xi$, where the direction ξ makes with the x-axis of the original coordinate system an angle φ determined by the formulas $\cos\varphi = \alpha/\sqrt{\alpha^2 + \beta^2}$, $\sin\varphi = \beta/\sqrt{\alpha^2 + \beta^2}$, and to repeat the same arguments as in the proof of the analogous inequality given in (2) for the operator δu.

Consider in some finite domain Ω of the (xy)-plane the nonhomogeneous weakly elliptic equation

$$Au + bu \equiv \sum_{i,k=0}^{l} (-1)^i \frac{\partial^l}{\partial x^{l-i} \partial y^i}\left(a_{ik}\frac{\partial^l u}{\partial x^{l-k} \partial y^k}\right) + bu \equiv f(x,y), \qquad l \geq 1,$$

$$(a_{ik} = a_{ki}). \tag{3}$$

Let the solution of this equation satisfy on the boundary S of the domain Ω the boundary conditions

$$u\Big|_S = 0, \qquad \frac{\partial u}{\partial x}\Big|_S = 0, \qquad \frac{\partial u}{\partial y}\Big|_S = 0, \dots, \qquad \frac{\partial^{l-1} u}{\partial x^{l-k-1}\partial y^k}\Big|_S = 0, \tag{4}$$

$$k = 0, 1, \dots, l-1$$

(the fundamental boundary-value problem). In what follows we shall assume that the following hypotheses are fulfilled: 1) the domain Ω is bounded by a piecewise smooth curve S with a center of symmetry; 2) the functions $a_{ik}(x, y)$ are continuously differentiable l times, while $b(x, y)$ is continuous and nonnegative in $\overline{\Omega}$; 3) the function $f(x, y)$ is continuous in $\overline{\Omega}$.

Using inequality (2), we shall, just as in (2), establish the unique solvability of the boundary-value problem posed and the convergence of the Ritz and Galerkin methods for this boundary-value problem. At the same time, the weakly elliptic equation of even order $2l$ considered in (2) is a rather special case of equation (3), when $\alpha = \beta = 1$.

2. Let us now introduce, in addition to the original operator $\delta_{\alpha,\beta} u = \alpha\, \partial u/\partial x + \beta\, \partial u/\partial y$, the operator $\bar{\delta}_{\alpha,\beta} u = \alpha\, \partial u/\partial x - \beta\, \partial u/\partial y$, conjugate* to it, and put

$$^*\Delta^l_{\alpha,\beta} u = \frac{1}{2}\left(\delta^{2l}_{\alpha,\beta} u + \bar{\delta}^{2l}_{\alpha,\beta} u\right) = \sum_{k=0}^{l} C^{2k}_{2l} \alpha^{2(l-k)} \beta^{2k} \frac{\partial^{2l} u}{\partial x^{2(l-k)} \partial y^{2k}}. \tag{5}$$

For $l = 1$ and $\alpha = \beta = 1$, operator (5) coincides with the Laplace operator $\Delta u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$. Further, every elliptic operator of second order of the form $A' u = a\, \partial^2 u/\partial x^2 + b\, \partial^2 u/\partial y^2$ with constant coefficients is also an operator $\Delta_{\alpha,\beta} u$, as is easy to verify by setting $\alpha = \sqrt{a}$, $\beta = \sqrt{b}$. If $l > 1$, then operator (5) contains the same derivatives as the polyharmonic operator $\Delta^l u$, and differs from the latter only by positive numerical coefficients at the corresponding deriv—

* We borrow this term from algebra.

ones. However, the polyharmonic operator $\Delta^l u$ has a different structure and is not reducible to the form of the operator $\overset{*}{\Delta}{}^l_{\alpha,\beta} u$. We shall call the operator $\overset{*}{\Delta}{}^l_{\alpha,\beta} u$ the **generalized Laplace operator**.

It is not difficult to verify that the generalized Laplace operator is elliptic and, consequently, also a weakly elliptic operator. In view of this, everything stated concerning the basic boundary-value problem for the weakly elliptic equation (3) carries over also to the same boundary-value problem for the nonhomogeneous generalized Laplace equation

$$(-1)^l \overset{*}{\Delta}{}^l_{\alpha,\beta} u = f(x,y), \qquad l \geqslant 0, \qquad f(x,y) \text{ is continuous in } \overline{\Omega}. \tag{6}$$

The homogeneous generalized Laplace equation $\overset{*}{\Delta}{}^l_{\alpha,\gamma} u = 0$ is the Euler–Ostrogradsky equation for the functional

$$\overset{*}{D}(u) = \iint_\Omega \sum_{k=0}^{l} C_{2l}^{2k} \alpha^{2(l-k)} \beta^{2k} \left(\frac{\partial^l u}{\partial x^{l-k} \partial y^k} \right)^2 d\Omega.$$

Further, the norm $\overset{*}{D}(u) = \|u\|^2_{L^{(l)}2}$ contains the squares of the same derivatives as does the norm $D(u) = \|u\|^2_{L^{(l)}2}$ in the case of the polyharmonic operator $\Delta^l u$, and differs from the latter only by the corresponding positive coefficients, independent of the choice of the function $u(x,y)$; consequently, the norms $D(u)$ and $\overset{*}{D}(u)$, for fixed α and β, are equivalent. In view of what has been said, the investigations carried out by S. L. Sobolev in the monograph ([1]) by the variational method in the case of the basic boundary-value problem for the homogeneous polyharmonic equation (as applied to functions of two variables) carry over also to the basic boundary-value problem for the homogeneous generalized Laplace equation.

Starting from the operator $\overset{*}{\Delta}{}^l_{\alpha,\beta} u$, one can also construct the generalized polyharmonic operator $(\overset{*}{\Delta}{}^l_{\alpha,\beta})^n u$–the n-fold application of the generalized Laplace operator. The generalized polyharmonic operator may, in our opinion, become the object of various investigations, in particular those that have already been applied to the ordinary polyharmonic operator.

3. Let us present still another new method (algorithm) for finding the exact solution of the nonhomogeneous generalized Laplace equation in the case of the basic boundary-value problem. This method is as follows.

Suppose it is required to find, in a finite domain Ω with smooth boundary S of the kind indicated above, a solution of equation (6) satisfying the boundary conditions (4). With the operators $\delta_{\alpha,\beta} u$ and $\overline{\delta}_{\alpha,\beta} u$ we introduce directions ξ and η, which we take as new coordinate axes with the old origin at the center of the curve S. The new coordinate system $\xi\eta$, being, generally speaking, oblique-angled, will be related to the original rectangular system xy by the formulas $x = \xi \cos\varphi + \eta \cos\psi$, $y = \xi \sin\varphi + \eta \sin\psi$, where φ and ψ are the angles made by the axes ξ and η with the x-axis of the original system, and the angle ψ is such that $\cos\psi = \cos\varphi$, $\sin\psi = -\sin\varphi$. If, in particular, $\alpha = \beta$, then $\varphi = \pi/4$, $\psi = -\pi/4$; consequently, the new system $\xi\eta$ will also be rectangular. Next, we transform the function $f(x, y)$ to the coordinate system $\xi\eta$ and integrate two ordinary differential equations

$$(-1)^l \frac{d^{2l}u}{d\xi^{2l}} = \frac{f(\xi,\eta)}{(\alpha^2+\beta^2)^l}, \qquad (-1)^l \frac{d^{2l}u}{d\eta^{2l}} = \frac{f(\xi,\eta)}{(\alpha^2+\beta^2)^l}.$$

Here the arbitrary constants of integration must be regarded as arbitrary functions of the other coordinate, entering the equation as a parameter. These arbitrary functions are found respectively

under the boundary conditions

$$u\Big|_S = 0, \quad \frac{du}{d\xi}\Big|_S = 0, \dots, \quad \frac{d^{\,l-1}u}{d\xi^{\,l-1}}\Big|_S = 0; \qquad u\Big|_S = 0, \quad \frac{du}{d\eta}\Big|_S = 0, \dots, \quad \frac{d^{\,l-1}u}{d\eta^{\,l-1}}\Big|_S = 0,$$

equivalent to conditions (4).

Let us denote the solutions of equations (7), respectively, by $u_0(\xi,\eta)$ and $\bar{u}_0(\xi,\eta)$, and the desired solution of equation (6) (in the system $\xi\eta$) by $u_*(\xi,\eta)$. In addition, put

$$g_0(\xi,\eta) = \tfrac{1}{2}(\alpha^2+\beta^2)^l \left(\frac{\partial^{2l}\bar{u}_0}{\partial\xi^{2l}} + \frac{\partial^{2l}u_0}{\partial\eta^{2l}} \right),$$

then the function $v_0 = u_0 + \bar{u}_0$ will satisfy the identity $\overset{*}{\Delta}{}^l_{\alpha,\beta} v_0 = f + g_0$. If it turns out that $g_0 = \lambda_0 f$, $\lambda_0 = \text{const}$, then

$$u_*(\xi,\eta) = \frac{1}{\lambda_0+1}\, v_0(\xi,\eta).$$

Otherwise, representing $g_0(\xi,\eta)$ in the form $g_0 = \bar{\lambda}_0 f + f_1$ (here it may be that $\bar{\lambda}_0 = 0$), we integrate, in an analogous way, equations (7) with the function $f(\xi,\eta)$ on the right-hand sides replaced by $f_1(\xi,\eta)$. As a result we obtain the function $v_1 = u_1 + \bar{u}_1$, satisfying the identity $\overset{*}{\Delta}{}^l_{\alpha,\beta} v_1 = f_1 + g_1$, where

$$g_1 = \tfrac{1}{2}(\alpha^2+\beta^2)^l \left(\frac{\partial^{2l}\bar{u}_1}{\partial\xi^{2l}} + \frac{\partial^{2l}u_1}{\partial\eta^{2l}} \right) = \bar{\lambda}_1 f_1 + f_2.$$

We shall consider this process completed if we obtain such a function $v_n = u_n + \bar{u}_n$ that satisfies the identity $\overset{*}{\Delta}{}^l_{\alpha,\beta} v_n = f_n + g_n$, where $g_n(\xi,\eta)$ is already represented in the form of a linear combination of the functions $f, f_1, \dots, f_n$. Eliminating from the system of the above-formed $n+1$ identities the functions

$f_1, f_2, \ldots, f_n$, we obtain the solution $u_*(\xi, \eta)$ of equation (6) in the form of a certain linear combination of the found functions $v_0, v_1, \ldots, v_n$.

The algorithm presented may also be applied to finding the solution of the homogeneous equation $\overset{*}{\Delta}{}^{l}_{\alpha,\beta} u = 0$ under prescribed inhomogeneous boundary conditions of the form (4). For this purpose it is necessary first to transform, in the usual way, the given homogeneous equation into an inhomogeneous one, and the boundary conditions into homogeneous ones.

In conclusion we note that the study of weakly elliptic operators, in our opinion, would open broad possibilities for mathematical physics and other applied sciences adjacent to it. By introducing weakly elliptic operators into mathematical physics, it would be possible to generalize significantly some results already known for elliptic operators, extending them to broader classes of operators. In application to variational methods, such a generalization is given in the present article. In addition, the introduction of weakly elliptic operators would facilitate the simplification of certain questions of mathematical physics, since these questions could be developed by the methods of the theory of ordinary linear differential equations. In confirmation of this, we note that the algorithm given above for finding the exact solution of equation (6) is obtained by applying only elementary methods of the theory of ordinary linear equations. Yet the method considered in classical mathematical physics for finding the exact solution of the Dirichlet problem for a disk (the Poisson integral), although it constitutes a rather narrow fact, is associated with considerable difficulties connected with the specificity of the classical methods of the theory of elliptic equations. At the same time, our algorithm is applicable to finding the exact solution for a broad class of equations of the form (6) in various practically most common domains, and in its structure is quite simple.

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REFERENCES CITED

1. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.

2. A. E. Martynyuk, DAN, **129**, No. 6 (1959).

Note: Figure translations are in progress. See original paper for figures.