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Abstract

Full Text

MATHEMATICS

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SEVERAL RESULTS ON CHARACTERISTIC FUNCTIONALS

(Presented by Academician A. N. Kolmogorov on 18 XI 1961)

Let X, Y be a pair of vector spaces put in duality by the bilinear form $\langle \cdot, \cdot \rangle$, and let $\mathcal{L} = \mathcal{L}_{X, Y}$ be the smallest σ -algebra of subsets of X with respect to which all linear forms $\langle \cdot, y \rangle$, $y \in Y$, are measurable. The characteristic functional (c.f.) of the distribution of a probability measure P , defined on \mathcal{L} , is called

$$\chi(y) = \int_X e^{i\langle x, y \rangle} P(dx).$$

1. A. N. Kolmogorov in ⁽¹⁾ indicated a generalization to c.f.'s of S. Bochner's theorem on necessary and sufficient conditions for a function to be characteristic, for the case of distributions in the space X conjugate to a countably Hilbert space Y (of countable type). This generalization turns out also to be valid for a somewhat broader class of spaces: it is sufficient to assume that X is conjugate to a separable countably pre-Hilbert* space Y , not necessarily being a space of countable type.
2. In the theorem referred to in item 1, the necessary and sufficient conditions on the functional consist of: 1) taking the value one at zero; 2) nonnegative definiteness; 3) continuity in the so-called J -topology. This J -topology may be defined as follows. Let Y be a separable countably pre-Hilbert space, $[\cdot, \cdot]_n$, $n = 1, 2, \dots$, a system of pseudoscalar products defining its topology, and \dot{X}_n the set of linear functionals continuous with respect to the topology generated by the n -th product. We denote by $(\cdot, \cdot)'_n$ the scalar product in \dot{X}_n defined in the usual way. The topology J is the topology generated by the system of pseudonorms of the form

$$P_{n, \mu}(y) = \left\{ \int_{\dot{X}_n} \langle x, y \rangle^2 \mu(dx) \right\}^{1/2},$$

where μ is a measure on the Borel sets in \dot{X}_n , for which

$$\int_{\dot{X}_n} (x, x)'_n \mu(dx) < \infty.$$

Such a definition of the J -topology makes it possible to define its analogue in an arbitrary locally convex metrizable space Y . It is natural to ask whether this analogue is suitable for generalizing Bochner's theorem. One can construct an example (namely, take $Y = l^p$, $1 < p < 2$) showing that the answer to this question is negative.

3. It is well known that, for compactness of a family of distributions in a finite-dimensional Euclidean space with respect to the weak topology

* A linear topological space is called countably pre-Hilbert if its topology is generated by a countable system of pseudoscalar products. (A pseudoscalar product differs from a scalar product in that in it the product of a nonzero element by itself may turn out to be equal to zero.)

necessary and sufficient is the equicontinuity of the corresponding characteristic functions at zero. In the present section we shall consider the question of generalizing this result to characteristic functionals.

Let Y be a separable countably pre-Hilbert space, X the space conjugate to it, and T_c the topology in X of uniform convergence on compact subsets of Y . We shall denote by w the (weak) topology in the space of distributions \mathfrak{B} on $\mathcal{L}_{X,Y}$ (here $\mathcal{L}_{X,Y}$ coincides with \mathfrak{B}_{T_c} , the smallest σ -algebra with respect to which all functions on X continuous relative to T_c are measurable), in which a base of neighborhoods of a point $P_0 \in \mathfrak{B}$ is formed by the sets

$$U_{f_1, \dots, f_n; \varepsilon}(P_0) = \left\{ P : \left| \int_X f_k d(P - P_0) \right| < \varepsilon, k = 1, \dots, n \right\},$$

where $\varepsilon > 0$ and f_1, \dots, f_n are bounded functions continuous on X relative to T_c .

Let Y be of countable type.

Theorem. *If in the space Y there exists a linear topology T such that relative sequential compactness in the topology w of a family of distributions on \mathfrak{B}_{T_c} in (X, T_c) is equivalent to the equicontinuity of the corresponding families of characteristic functionals at zero in Y in the topology T , then Y is nuclear. In a nuclear space Y such a topology exists—as T one may take simply the original topology.*

Without the assumption that Y is of countable type, the assertion obtained from this theorem by replacing the words “relative sequential compactness in the topology w ” by the word “tightness” is true. (Recall that a family of distributions is tight if there exist compact sets with arbitrarily small probability of their complements, uniformly over all distributions of the family.)

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REFERENCES

1. A. N. Kolmogorov, *Theory of Probability and Its Applications*, **4**, 2, 237 (1959).

Note: Figure translations are in progress. See original paper for figures.

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