# Reports of the Academy of Sciences of the USSR

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1962

SovietRxiv

## Abstract

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Reports of the Academy of Sciences of the USSR

1962. Volume 142, No. 5

## MATHEMATICS

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# ON ENTIRE AND MEROMORPHIC SOLUTIONS OF ONE CLASS OF DIFFERENCE EQUATIONS

*(Presented by Academician I. G. Petrovskii on 20 X 1961)*

Equations of the form are considered

$$L(y) \equiv \sum_{k=0}^{m} (a_k + b_{kx}) y(x + h_k) = f(x), \tag{1}$$

where

$$0 = h_0 < h_1 < \cdots < h_m, \qquad b_0 \neq 0, \qquad b_m \neq 0. \tag{2}$$

Equations of this and of a more general type, differential-difference equations, have been the subject of study in a number of works ($^{1-4}$). Considering equation (1) as a special case of a differential equation of infinite order, we can apply to it the results of works ($^{5-7}$). In the works listed, for equation (1) solutions were constructed both regular in some domain and regular in the whole plane. But when entire solutions were involved, these were always only solutions of growth not exceeding exponential growth.

In the present work, by the matrix method ($^{7}$), conditions are established for the solvability of equation (1) in the class of entire functions for an arbitrary entire right-hand side. An estimate is given for the growth of an entire solution in the case when the right-hand side is an entire function of growth $[r, \sigma]$, $r > 1$. The maximal growth of an entire solution of the homogeneous equation is determined. The results obtained make it possible to apply to equation (1) the method developed by A. G. Naftalevich for a difference equation with constant coefficients ($^{8}$). Using this method, for an entire and for a meromorphic right-hand side we construct a meromorphic solution having, in some vertical strip, arbitrarily prescribed poles with arbitrarily prescribed principal parts.

Equation (1) can be written in another way:

$$L(y) \equiv \sum_{s=0}^{\infty} (A_s + B_{sx}) y^{(s)} = f(x). \tag{1'}$$

In this case we shall have

$$a(t) = \sum_{s=0}^{\infty} A_{st}^{s} = \sum_{k=0}^{m} a_{ke}^{h_{kt}}, \qquad b(t) = \sum_{s=0}^{\infty} B_{st}^{s} = \sum_{k=0}^{m} b_{ke}^{h_{kt}}.$$

Let $f(x)$ be an entire function. We shall find an entire solution of equation $(1')$. Substituting into $(1')$ the expansions $y(x) = \sum_{n=0}^{\infty} c_{nx}^{n}$ and $f(x) = \sum_{n=0}^{\infty} d_{nx}^{n}$, we arrive at the system

$$Mc = d, \tag{3}$$

where $\lim_{n\to\infty} \sqrt[n]{|d_n|} = 0$, and the solution is sought under the condition $\lim_{n\to\infty} \sqrt[n]{|c_n|} = 0$. Expressing ourselves in the language of analytic spaces (9), we must solve system (3) in the space $A_\infty$ with right-hand side from $A_\infty$. Let us pass to the system, deter-

by the matrix $M^*$, transposed to $M$,

$$M^* \tilde{c} = \tilde{d}. \tag{4}$$

We must solve system (4) in the space $\overline{A}_0$ with right-hand side from $\overline{A}_0$, i.e., with right-hand side $\tilde{d}$ for which $\overline{\lim}_{n\to\infty} \sqrt[n]{|\tilde{d}_n|} < \infty$, a solution $\tilde{c}$ is sought for which $\overline{\lim}_{n\to\infty} \sqrt[n]{|\tilde{c}_n|} < \infty$. System (4) is equivalent to the equation

$$\tilde{L}(z) \equiv a(x)z + b(x)z' = g(x), \tag{5}$$

where

$$z(x) = \sum_{n=0}^{\infty} \frac{\tilde{c}_n}{n!} x^n, \qquad g(x) = \sum_{n=0}^{\infty} \frac{\tilde{d}_n}{n!} x^n. \tag{6}$$

We need to find a solution of equation (5), in the class of entire functions of growth not exceeding exponential, for a right-hand side that is an entire function also of growth not exceeding exponential (no more precise growth conditions are imposed).

Let us show that if the function

$$z(x) = C \exp\left[-\int_{x_0}^{x} \frac{a(x)}{b(x)}\,dx\right] + \exp\left[-\int_{x_0}^{x} \frac{a(x)}{b(x)}\,dx\right] \int_{x_0}^{x} \frac{g(x)}{b(x)} \exp\left[\int_{x_0}^{x} \frac{a(x)}{b(x)}\,dx\right] dx =$$

$$= C z_1(x) + z_2(x) \tag{7}$$

is entire, then its growth is not higher than exponential. Indeed, under the fulfillment of conditions (2), outside a certain vertical strip $|\operatorname{Re} x| < p$ we shall have

$$\left|\frac{a(x)}{b(x)}\right| < M. \tag{8}$$

It is known that inside this vertical strip there is a sequence of horizontal bridges, determined by the relations

$$\operatorname{Im} x = q_n, \qquad q_n - q_{n-1} < q,$$

such that on these bridges $|b(x)| > N$, i.e., (8) is also valid. Consequently, on the bridges and outside the strip,

$$\int_{x_0}^{x} \frac{a(x)}{b(x)}\,dx = O(x).$$

If $g(x)$ is an entire function of growth not exceeding exponential, then outside the strip and on the bridges the fraction $a(x)/b(x)$ will grow no faster than a certain exponential. Hence, if $z(x)$ is entire, then outside the strip and on the bridges its growth is not higher than exponential. This also implies our assertion.

The investigation of the conditions under which the function $z(x)$ will be entire, and the subsequent use of the complete classification of infinite matrices (7), leads to the following assertion.

**Theorem 1.** *Equation (1) has an entire solution for every entire right-hand side $f(x)$ if and only if among the zeros of the function $b(x)$ there is at least one at which the fraction $a(x)/b(x)$ has either a multiple pole, or a simple pole with residue different from a negative integer. In the case, and only in the case, when all zeros of $b(x)$, with the possible exception of a finite number, are simple and the fraction $a(x)/b(x)$ has at them poles with residues different from negative integers, the solution contains a finite number of arbitrary constants; in the remaining cases there will be an infinite set of arbitrary constants.*

Equation (1) will have a unique entire solution for any entire right-hand side if and only if all the zeros of $b(x)$ are simple and at each of them the fraction $a(x)/b(x)$ has a pole with residue different from a negative integer.

If, however, at any zero of $b(x)$ the fraction $a(x)/b(x)$ is either regular or has a simple pole with residue equal to a negative integer, then equation (1) has an entire solution if and only if $f(x)$ belongs to the set of functions orthogonal to the function associated with $z_1(x)$. In this case the solution depends on an infinite set of arbitrary constants.

We now substitute into equation (1′) the expansions

$$y(x) = \sum_{k=0}^{\infty} \frac{c_k}{(k!)^{\alpha}} x^k, \qquad f(x) = \sum_{k=0}^{\infty} \frac{d_k}{(k!)^{\beta}} x^k.$$

This will lead us to the system:

$$P_{\beta}^{r} M P_{-\alpha} c = d, \tag{9}$$

where $P_s$ is a matrix whose main diagonal contains the elements $1, 1!^s, (2!)^s, (3!)^s, \ldots$, while all the other elements are zero. Similarly, substituting into equation (5) the expansions

$$z(x) = \sum_{k=0}^{\infty} \frac{\tilde{c}_k}{(k!)^{\alpha_1}} x^k, \qquad g(x) = \sum_{k=0}^{\infty} \frac{\tilde{d}_k}{(k!)^{\beta_1}} x^k$$

leads to the system

$$P_{\beta_1 - 1} M^* P_{1-\alpha_1} \tilde{c} = \tilde{d}. \tag{10}$$

Suppose that in equation (5) the right-hand side $g(x)$ is an entire function of growth $[\rho, \sigma]$, where $\rho > 1$. Obviously, in this case the growth of the entire solution of equation (5) cannot be less than $[\rho, \sigma]$. But, on the other hand, from formula (7), by arguments analogous to those given above, it is easy to see that for $\rho > 1$ (and when conditions (2) are satisfied) the growth of the entire solution is not higher than $[\rho, \sigma]$. Consequently, the operator $\widetilde{L}(z)$ transforms any entire function of growth $[\rho, \sigma]$, $\rho > 1$, into an entire function of exactly the same growth. In other words, for

$$\alpha_1 = \frac{1}{\rho}, \quad \rho > 1, \quad \overline{\lim_{n\to\infty}} \sqrt[n]{|\tilde{c}_n|} = \sigma$$

we have

$$\beta_1 = \frac{1}{\rho}, \quad \overline{\lim_{n\to\infty}} \sqrt[n]{|\tilde{d}_n|} = \sigma$$

(and conversely). The system transposed to (10), i.e. system (9), must have the form

$$P_{1-1/\rho} M P_{1/\rho-1} c = d. \tag{11}$$

For $\rho > 1$ the matrix $P_{1-1/\rho} M P_{1/\rho-1}$ maps the space $\overline{A}_\sigma$ into itself for any $\sigma$. M. G. Khaplanov showed that in this case, for any point $c$, we shall have

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} = \overline{\lim_{n\to\infty}} \sqrt[n]{|d_n|}.$$

Comparing systems (9) and (11), we obtain

$$\alpha = \beta = \frac{\rho - 1}{\rho}.$$

We note that when $\rho$ runs through all possible values greater than one, the fraction

$$\frac{\rho}{\rho - 1} = r$$

also runs through all possible values greater than one, and conversely.

**Theorem 2.** If in equation (1) the right-hand side $f(x)$ is an entire function of growth $[r, \sigma]$, $r > 1$, then the entire solution of this equation (in the case of its existence) will also have growth $[r, \sigma]$.

**Theorem 3.** If the right-hand side $f(x)$ of equation (1) is an entire function of growth not exceeding $[1, \sigma]$, $\sigma \leqslant \infty$ (in particular, $f(x) \equiv 0$), then such an equation has no entire solutions of growth greater than $[1, \infty]$.

Now, having constructed entire solutions, we can construct meromorphic solutions of equation (1). We formulate the results.

Let $\pi$ be an arbitrary vertical half-closed strip of width $h_m$, $a \leqslant \operatorname{Re} x < a + h_m$. To a point $\lambda \in \pi$, $a + h_s \leqslant \operatorname{Re} \lambda < a + h_{s+1}$, we assign two sets: the set $\Delta$, consisting of the points

$$\lambda_{k_1 \ldots k_n} = \lambda - h_m + \sum_{\nu=1}^{n} (h_m - h_{k_\nu}),$$

and the set $\delta$, consisting of the points

$$\mu_{l_1 \dots l_n} = \lambda - \sum_{\nu=1}^{n} h_{l_\nu},$$

$(n = 1, 2, \dots; \quad k_1 = 1, 2, \dots, s; \quad l_1 = s{+}1, \dots, m; \quad k_2, \dots, k_n = 1, 2, \dots, m{-}1; \quad l_2, \dots, l_n = 2, 3, \dots, m).$

**Theorem 4.** Suppose that in equation (1) the right-hand side $f(x)$ is an entire function (in particular, $f(x) \equiv 0$) and that among the zeros of $b(x)$ there is at least one such zero at which the fraction $a(x)/b(x)$ has a pole, either multiple or simple with residue different from a negative integer. In the strip $\pi$, prescribe arbitrarily a sequence $\{\lambda^{(i)}\}$ having no points of accumulation at finite distance and such that no point of the set $\bigcup_i \Delta^{(i)}$ coincides with the point $x = -a_m/b_m$ and no point of the set $\bigcup_i \delta^{(i)}$ coincides with the point $x = -a_0/b_0$. In this case, for equation (1) one can construct a meromorphic solution $y(x)$ having in the strip $\pi$ poles at the points $\{\lambda^{(i)}\}$ and only at them, with arbitrarily prescribed principal parts.

Denote by $\alpha^{(\nu)}$, $\operatorname{Re} \alpha^{(\nu)} \geqslant a$; $\beta^{(\tilde{\nu})}$, $\operatorname{Re} \beta^{(\tilde{\nu})} < a$ $(\nu, \tilde{\nu} = 1, 2, \dots)$, all those points where the poles of the meromorphic function $\varphi(x)$ are located. Put

$$\alpha^{(\nu)}_{k_1 \dots k_n} = \alpha^{(\nu)} + \sum_{j=1}^{n} (h_m - h_{k_j}), \qquad \beta^{(\tilde{\nu})}_{l_1 \dots l_n} = \beta^{(\tilde{\nu})} - \sum_{j=1}^{n} h_{l_j}.$$

$$(n = 1, 2, \dots; \quad k_1, \dots, k_n = 1, 2, \dots, m-1; \quad l_1, \dots, l_q = 2, 3, \dots, m).$$

**Theorem 5.** If to the conditions of Theorem 4 one adds the requirement that none of the points $\alpha^{(\nu)}$, $\alpha^{(\nu)}_{k_1, \dots, k_n}$ coincide with the point $x = -a_m/b_m$, and none of the points $\beta^{(\tilde{\nu})}$, $\beta^{(\tilde{\nu})}_{l_1 \dots l_n}$ coincide with the point $x = -a_0/b_0$, then also for the equation $L(y) = \varphi(x)$ one can assert the existence of a meromorphic solution having in the strip $\pi$ poles at the points $\{\lambda^{(i)}\}$ and only at them, with arbitrarily prescribed principal parts.

In conclusion I express my gratitude to my scientific adviser M. G. Khaplanov.

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Received
13 X 1961

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*Note: Figure translations are in progress. See original paper for figures.*

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