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Abstract

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ON THE POINCARÉ PROBLEM FOR A SECOND-ORDER ELLIPTIC EQUATION WITH SINGULAR COEFFICIENTS

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In the present note we study the plane boundary-value Poincaré problem (problem A) for an elliptic equation of the form

$$\Delta u + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu = 0 \tag{1}$$

in the case when the coefficients $a(x, y)$, $b(x, y)$ have a polar singularity of the first order, and the coefficient $c(x, y)$ has a polar singularity of the second order at some fixed point of the domain*. In the case when $a, b, c \in L_p$ for $p > 2$, this problem, both for simply connected and for multiply connected domains, has been well studied ($^{1-3}$). We also note that if $c \equiv 0$, then equation (1), with respect to the function $\partial u/\partial x - i\,\partial u/\partial y$, is transformed into a generalized Cauchy–Riemann complex equation (^{1,4}).

1. Denote by D a bounded simply connected domain in the plane of the variables x, y, with boundary Γ of class (^{i}) C_α, $0 < \alpha < 1$. Let $S_\varepsilon(D)$ be the Banach space of functions $f(z)$ representable in the form

$$f(z) = \frac{f_0(z)}{(x^2 + y^2)^{\varepsilon/2}}, \qquad f_0(z) \in S(D),$$

with norm

$$\|f\|_{S_\varepsilon(D)} = \sup_{x,y\in D} (x^2 + y^2)^{\varepsilon/2}|f(z)|,$$

where $S(D)$ is the space of bounded measurable functions f_0 with norm

$$\sup_{x,y\in D} |f_0(z)|.$$

We shall say that $u \in S^1_\varepsilon(D)$ if u has a generalized derivative, $u \in S_\varepsilon(D)$, and

$$\frac{\partial u}{\partial z} \in S_{1+\varepsilon}(D),$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

Problem A. It is required to find a real-valued function $u \in S^1_\varepsilon(D)$ satisfying in D the equation

$$\Delta u + \frac{a}{r}\frac{\partial u}{\partial x} + \frac{b}{r}\frac{\partial u}{\partial y} + \frac{c}{r^2}u = 0 \tag{1'}$$

and on Γ the condition

$$\alpha u_x + \beta u_y = h, \tag{2}$$

where a, b, c are given real bounded measurable functions in D; α, β, h are real functions given on Γ, Hölder continuous, and

$$r = \sqrt{x^2 + y^2}.$$

Below, in order to study problem A, we shall relate it to a certain auxiliary Riemann–Hilbert boundary-value problem (problem P) for an elliptic system in two complex-valued functions, with coefficients having a polar singularity of the first order.

Introducing the operator

$$\frac{\partial}{\partial \bar z} = \overline{\frac{\partial}{\partial z}},$$

where the bar above indicates passage to the complex conjugate expression, problem A can be written in the form:

$$\frac{\partial^2 u}{\partial \bar z\, \partial z} + \frac{A(z)}{|z|}\frac{\partial u}{\partial z} + \frac{\overline{A(z)}}{|z|}\frac{\partial u}{\partial \bar z} + \frac{c(x,y)}{4|z|^2}u = 0; \tag{1''}$$

$$\operatorname{Re} \lambda(t)\frac{\partial u}{\partial t} = h(t), \tag{2'}$$

where

$$A(z) = (a + ib)/4, \qquad \alpha(t) + i\beta(t) = \lambda(t).$$

[*] This point can always be shifted to the origin of coordinates.

Let $z = \varphi(\zeta)$ be a holomorphic function conformally mapping the domain D onto the unit disk and satisfying the conditions

$$\varphi(0) = 0, \qquad \varphi'(0) > 0. \tag{3}$$

It follows from (3) that

$$\varphi(\zeta) = \zeta\varphi_0(\zeta), \qquad \varphi_0(\zeta) \neq 0 \text{ anywhere.} \tag{4}$$

By virtue of (4) it is not difficult to see that problem A is conformally invariant. Therefore, everywhere below, without restricting the generality of the problem, we shall assume that D is the unit disk: $|z| < 1$.

In this case equation (1) and condition (2) may be written in the form:

$$\frac{\partial}{\partial \bar{z}}\left(z\frac{\partial u}{\partial z}\right) + \frac{A(z)}{|z|}\left(z\frac{\partial u}{\partial z}\right) + \frac{\overline{A(z)}}{|z|}\frac{z}{\bar{z}}\overline{\left(z\frac{\partial u}{\partial z}\right)} + \frac{c}{4z}u = 0; \tag{1”’}$$

$$\operatorname{Re} \bar{t}\cdot\lambda(t)\left(t\frac{\partial u}{\partial t}\right) = h(t). \tag{2”}$$

In what follows, an essential role is played by the study of the following problem:

Problem P. It is required to find a pair of functions $u_1(z), u_2(z) \in S_\varepsilon(D)$, satisfying in D the system of equations

$$\frac{\partial u_1}{\partial \bar{z}} = \frac{1}{\bar{z}}\bar{u}_2, \qquad \frac{\partial u_2}{\partial \bar{z}} + \frac{A(z)}{|z|}u_2 + \frac{\overline{A(z)}}{|z|}\frac{z}{\bar{z}}\bar{u}_2 + \frac{c}{4z}u_1 = 0 \tag{5}$$

and on Γ the conditions

$$\operatorname{Re} iu_1 = 0, \qquad \operatorname{Re} \bar{t}\cdot\lambda(t)u_2 = h(t). \tag{6}$$

The connection between problems A and P is established by the following:

Theorem 1. If $u(x,y) \in S^1_\varepsilon(D)$ is a solution of problem A, then the pair of functions

$$u_1 = u, \qquad u_2 = z\frac{\partial u}{\partial z}$$

of class $S_\varepsilon(D)$ is a solution of problem P.

Conversely, if a pair of functions u_1, u_2 of class $S_\varepsilon(D)$ is a solution of problem P and if, in addition, the homogeneous Dirichlet problem

$$\Delta u + \frac{c(x,y)}{r^2}u = 0, \qquad u\big|_\Gamma = 0 \tag{7}$$

has only the trivial solution from $S^1_\varepsilon(D)$, then the function u_1 belongs to the class $S^1_\varepsilon(D)$ and is a solution of problem A.

Denote by l_P the number of linearly independent (over the field of real numbers) solutions of the homogeneous problem P^0 $(h \equiv 0)$, and by l_A the number of linearly independent solutions of the corresponding homogeneous problem A^0 $(h \equiv 0)$. Let, in addition, l_D be the number of linearly independent solutions of the Dirichlet problem (7). Then the relation ([2,3]) holds:

$$l_P - l_A = q, \tag{8}$$

where q is an integer nonnegative number not exceeding l_D.

2. Suppose that

$$\chi = \frac{1}{2\pi}\{\arg \lambda(t)\}_\Gamma \geq -1.$$

Then, proceeding in the same way as in ([1,3]), we verify that problem P is equivalent to the following system of integral equations:

$$u_1 - T_1 u_2 = \tilde{c}_1; \tag{9}$$

$$u_2 - T_2 u_2 - T_3 u_1 = \Phi'(z), \tag{10}$$

where T_1, T_2, T_3 are linear bounded operators in the Banach space $S_\varepsilon(\bar{D})$ (cf. (4)), which have the form

$$T_1 u_2 = -\frac{1}{\pi}\iint_D \left\{ \frac{e^{i\varphi(\zeta)}\overline{u_2(\zeta)}}{\bar{\zeta}(\zeta - z)} - \frac{z e^{-i\varphi(\zeta)} u_2(\zeta)}{\zeta(1 - \bar{\zeta} z)} \right\} d\xi\, d\eta; \tag{11}$$

$$T_2 u_2 = \frac{1}{\pi}\iint_D \left\{ \frac{A(\zeta)u_2 + \overline{A(\zeta)}\dfrac{\zeta e^{i\varphi(\zeta)}}{\zeta e^{i\varphi(\zeta)}}\bar{u}_2}{|\zeta|(\zeta - z)} + z^{2\chi+3}\frac{\overline{A(\zeta)}u_2 + A(\zeta)\dfrac{\overline{\zeta e^{i\varphi}}}{\zeta e^{i\varphi}}\bar{u}_2}{|\zeta|(1 - \bar{\zeta} z)} \right\} d\xi\, d\eta; \tag{12}$$

$$T_3 u_1 = \frac{1}{4\pi}\iint_D \left\{ \frac{e^{i\varphi(\zeta)}\overline{u_1(\zeta)}}{\zeta(\zeta - z)} + z^{2\chi+3}\frac{e^{-i\varphi(\zeta)}u_1}{\zeta(1 - \bar{\zeta} z)} \right\} c(\xi, \eta)\, d\xi\, d\eta, \tag{13}$$

where $\varphi(z)$ is a completely determined single-valued holomorphic function in D, and

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma h(t) e^{\Omega(t)} \frac{t + z}{t - z}\frac{dt}{t} + i\tilde{\alpha}_0 z^{\chi+1} +$$

$$+\sum_{k=0}^{\chi}\{\alpha_k(z^k - z^{2\chi+2-k}) + i\beta_k(z^k + z^{2\chi+2-k})\}. \tag{14}$$

Here $\Omega = \operatorname{Im}\varphi$; c_1, $\tilde{\alpha}_0$, α_k, β_k $(k = 0, 1, 2, \dots, \chi)$ are arbitrary real constants.

Let

$$K_{a,b} = \sup_{x,y\in D} \frac{r^\varepsilon}{2\pi} \iint_D \left(\frac{1}{|\zeta - z|} + \frac{r^{2\chi+3}}{|1 - \bar{\zeta}z|}\right) \sqrt{\frac{a^2 + b^2}{(\xi^2 + \eta^2)^{1+\varepsilon}}}\, d\xi\, d\eta; \tag{15}$$

$$K_{\nu,\chi} = \sup_{x,y\in D} \frac{r^\varepsilon}{\pi} \iint_D \left(\frac{1}{|\zeta - z|} + \frac{r^{2\chi+3}}{|1 - \bar{\zeta}z|}\right) |\nu(\xi, \eta)|\, d\xi\, d\eta. \tag{16}$$

Then, if the inequality

$$K_{a,b} + K_{1/4,-1} K_{c/4,\chi} < 1, \tag{17}$$

is satisfied, the system (9)—(10) is always uniquely solvable. The system (9)—(10) corresponding to the homogeneous problem P^0 $(h \equiv 0)$ then has exactly $2\chi + 4$ linearly independent (over the field of real numbers) solutions. Since every solution of problem (7) is also a solution of the problem

$$\Delta u + \frac{c}{r^2} u = 0, \qquad x'(s) u_x + y'(s) u = 0, \tag{18}$$

it is not difficult to see that, when inequality (17) is fulfilled, the number l_D does not exceed two. Then, taking into account relation (8), we arrive at the following result:

Theorem 2. If $\chi \geq -1$, then, when inequality (17) is fulfilled, problem A is always solvable, and the corresponding homogeneous problem A^0 $(h \equiv 0)$ has exactly $2\chi + 4 - q$ linearly independent solutions, where q is a nonnegative integer not exceeding two.

3. Let $\chi < -1$. In this case, following (1) (p. 298), put

$$i z^m u_1 = v_1, \qquad z^m u_2 = v_2, \qquad m = -(\chi + 1) > 0. \tag{19}$$

Then the functions v_1, v_2 will satisfy the problem

$$\frac{\partial v_1}{\partial \bar{z}} - \frac{\overline{e^{i\varphi(\zeta)}}}{z}\, i \left(\frac{z}{\bar{z}}\right)^m \bar{v}_2 = 0; \tag{20}$$

$$\frac{\partial v_2}{\partial \bar z} + \frac{A(z)}{|z|} v_2 + \frac{\overline{A(z)}}{|z|} \frac{z^{m+1} e^{i\varphi(\zeta)}}{\bar z^{m+1} e^{i\varphi(\zeta)}} \bar v_2 + i\, \frac{c(x,y) e^{i\varphi(\zeta)}}{4\bar z} \left(\frac{z}{\bar z}\right)^m \bar v_1 = 0, \tag{21}$$

$$\operatorname{Re} t^{-m} v_1 = 0, \qquad \operatorname{Re} v_2 = e^{\Omega} h.$$

Since $m > 0$, one can, as above, construct the corresponding equivalent system of integral equations with respect to the functions v_1 and v_2.

Returning then by formulas (19) to the functions u_1, u_2 (after transformations analogous to (1), (p. 298)) and taking into account that $u_1, u_2 \in S_\varepsilon(D)$, we obtain the system of integral equations

$$u_1 - \hat T_1 u_2 = P_{2m-1}(z); \tag{22}$$

$$u_2 - \hat T_2 u_2 - \hat T_3 u_1 = \frac{1}{\pi i} \int_\Gamma \frac{h(t) e^{\Omega(t)}\, dt}{t^m (t - z)}, \tag{23}$$

where

$$\hat T_1 u_2 = \frac{1}{\pi} \iint_D \left\{ \frac{i e^{i\varphi(\zeta)} \overline{u_2(\zeta)}}{\zeta(\zeta - z)} - z^{-2\varkappa-2} \frac{\bar\zeta^{-2\varkappa-3} i e^{-i\varphi(\zeta)} u_2(\zeta)}{\zeta(1 - \bar\zeta z)} \right\} d\xi\, d\eta, \tag{24}$$

$$\hat T_2 u_2 = \frac{1}{\pi} \iint_D \left\{ \frac{A(\zeta) u_2 + \overline{A(\zeta)} \dfrac{\zeta e^{i\varphi(\zeta)}}{\bar\zeta e^{i\varphi(\zeta)}} \overline{u_2(\zeta)}}{|\zeta|(\zeta - z)} + \bar\zeta^{-2\varkappa-3} \frac{\bar A u_2 + A \dfrac{\bar\zeta e^{i\varphi(\zeta)}}{\zeta e^{i\varphi(\zeta)}}}{|\zeta|(1 - \bar\zeta z)} \right\} d\xi\, d\eta; \tag{25}$$

$$\hat T_3 u_1 = \frac{1}{4\pi} \iint_D \left\{ \frac{i e^{i\varphi(\zeta)} \overline{u_1(\zeta)}}{\bar\zeta(\zeta - z)} - \frac{\bar\zeta^{-2\varkappa-3} e^{-i\varphi(\zeta)} u_1(\zeta)}{\zeta(1 - \bar\zeta z)} \right\} c(\xi, \eta)\, d\xi\, d\eta; \tag{26}$$

$$P_{2m-1}(z) = \hat\alpha_0 + \sum_{k=1}^{m} (\hat\beta_{m-k} + i\hat\alpha_{m-k}) z^k + \sum_{k=m}^{2m-2} \mu_k z^{k+1}; \tag{27}$$

$$\mu_k = \frac{1}{\pi} \iint_D \bar\zeta^{\,k} e^{-i\varphi(\zeta)} u_2(\zeta) \zeta^{-1}\, d\xi\, d\eta.$$

In order that the solution of the system (22)–(23) be a solution of problem (20)–(21), it is necessary and sufficient that l relations be satisfied. The number of these relations is determined by the inequality

$$-2\varkappa - 4 \leq l \leq -4(\varkappa + 1). \tag{28}$$

Thus we obtain the following result:

Theorem 3. Let $\varkappa < -1$. Then, if the inequality

$$\Lambda = \sup_{x,y \in D} \frac{r^{\varepsilon}}{2\pi} \iint_D \left\{ \frac{1}{|\zeta - z|} + \frac{|\zeta|^{-2\varkappa-3}}{|1 - \bar{\zeta} z|} \right\} \frac{\sqrt{a^2 + b^2} + \frac{1}{2}|c|\tilde{\Lambda}}{\sqrt{(\xi^2 + \eta^2)^{1+\varepsilon}}} \, d\xi \, d\eta < 1, \tag{29}$$

where

$$\tilde{\Lambda} = \sup_{x,y \in D} \frac{r^{\varepsilon}}{\pi} \iint_D \left\{ \frac{1}{|\zeta - z|} + \frac{r^{-2\varkappa-2}|\zeta|^{-2\varkappa-3}}{|1 - \bar{\zeta} z|} \right\} d\xi \, d\eta,$$

then for the solvability of problem A it is necessary and sufficient that a finite number of conditions be fulfilled.

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References

1. I. N. Vekua, *Generalized analytic functions*, Moscow, 1959.
2. Ten En-cher, Candidate dissertation, Siberian Branch of the Academy of Sciences of the USSR, 1960.
3. A. Dzhuraev, DAN, **142**, No. 5 (1962).
4. L. G. Mikhailov, DAN, **129**, No. 3 (1959).

Note: Figure translations are in progress. See original paper for figures.

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