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Abstract

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A CHARACTERIZATION OF THE RING OF ALL CONTINUOUS FUNCTIONS ON A BICOMPACTUM

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Let B be a commutative semisimple regular normed ring with identity, and let $\mathfrak{M}$ be the space of maximal ideals of the ring B. Using the Gelfand representation, the ring B can be realized as a ring of continuous functions on the bicompactum $\mathfrak{M}$. For every closed set $F \subset \mathfrak{M}$, by $I(F)$ we shall denote the closed ideal of the ring B formed from all functions $f \in B$ that vanish on F, and by B_F the quotient ring $B/I(F)$. In the paper (^{1}) Katznelson proved that if the ring B is symmetric and the idempotents* in each quotient ring B_F are bounded in the aggregate (by a constant which, generally speaking, depends on F), then $B = C(\mathfrak{M})$.

The assumption of the symmetry of B in Katznelson' s theorem is restrictive, since, on the one hand, there exist regular but nonsymmetric rings (^{2,3}), and, on the other hand, by the well-known Stone–Weierstrass theorem, every symmetric ring is dense in $C(\mathfrak{M})$.

Relying on the result of (^{4}), it is not difficult to prove that if B is a ring with uniform convergence, then the assumption of the symmetry of B in Katznelson' s theorem can be removed. It turned out that the same is true in the general case, namely, the following theorem holds.

Theorem 1. *If the idempotents in each quotient ring B_F are bounded, then $B = C(\mathfrak{M})$.*

The proof of this theorem can be carried out according to the scheme of the paper (^{1}). Let us first note that Lemmas 3, 4, 5 of that paper and the lemma from the paper (^{5}) used there are valid without the assumption of symmetry. In view of these results, it is enough to prove the theorem under the assumption that the idempotents in each quotient ring B_F are bounded by a constant K, common for all F.

We note that in the constructions given below we nowhere use the fact that elements of B can be multiplied; in other words, these constructions, with the appropriate qualifications, can be carried out for a normed space of functions.

Lemma 1. *Let the idempotents in each quotient ring B_F be bounded by a constant K independent of F.*

Then for any two disjoint closed sets $F_1, F_2 \subset \mathfrak{M}$ and arbitrary $\varepsilon > 0$, one can find a function $f \in B$ having the following properties:

1) $\|f\| \leq K$.
2)

$$f(M) = \begin{cases} 1, & \text{if } M \in F_1, \\ 0, & \text{if } M \in F_2; \end{cases}$$

3) $|\operatorname{Im} f(M)| \leq \varepsilon$;
4) $-\varepsilon \leq \operatorname{Re} f(M) \leq 1 + \varepsilon$.

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* An element h is called idempotent if $h^2 = h$.

To prove this lemma it is enough to show that for any three mutually disjoint closed sets F_1, F_2, F_3 there exists a function $g \in B$ for which:

$$\begin{aligned} &1') \quad \|g\| \leq K; \\ &2') \quad g(M) = \begin{cases} 1, & \text{if } M \in F_1, \\ 0, & \text{if } M \in F_2, \\ -1, & \text{if } M \in F_3; \end{cases} \\ &3') \quad |\operatorname{Im} g(M)| \leq \varepsilon; \\ &4') \quad |\operatorname{Re} g(M)| \leq 1 + \varepsilon. \end{aligned}$$

Doubling, if necessary, the constant K, we can always indicate a function $g_1 \in B$ satisfying conditions 1′) and 2′). Let

$$\Phi = \{M \in \mathfrak{M} : |\operatorname{Im} g_1(M)| \geq \varepsilon_1\};$$

$$\Psi = \{M \in \mathfrak{M} : |\operatorname{Re} g_1(M)| \geq 1 + \varepsilon_1\},$$

$$F_2' = F_2 \cup \Phi \cup \Psi.$$

Since

$$F_1 \cap F_2' = \emptyset, \qquad F_3 \cap F_2' = \emptyset,$$

we can find a function $\varphi \in B$ satisfying condition $1')$ and condition $2')$ for the sets F_1, F_2', F_3. Put

$$g_2 = \frac{1}{2}(g_1 + \varphi).$$

Then g_2 satisfies conditions $1')$ and $2')$, and also

$$|\operatorname{Im} g_2(M)| \leq \frac{K + \varepsilon_1}{2}, \qquad |\operatorname{Re} g_2(M)| \leq 1 + \frac{K - 1 + \varepsilon_1}{2}.$$

We note that such a function g_2 can be constructed for any of the three sets, and therefore, by repeating the same device several times, we arrive at a function satisfying all the necessary requirements.

The following lemma is entirely analogous to Lemma 1 from (1).

Lemma 2. *Suppose that there exist constants K and $0 < K_1 < 1$ such that for every function $f \in C(\mathfrak{M})$ there is an element $f_1 \in B$ such that*

$$\|f_1\| \leq K \sup_{M \in \mathfrak{M}} |f(M)|, \qquad \sup_{M \in \mathfrak{M}} |f(M) - f_1(M)| \leq K_1 \sup_{M \in \mathfrak{M}} |f(M)|.$$

Then $B = C(\mathfrak{M})$, and for every $f \in B$

$$\|f\| \leq 4K(1 - K_1)^{-1} \sup_{M \in \mathfrak{M}} |f(M)|.$$

The definitions given below and Lemma 3 are a forced formalization of a very simple geometric fact. Let S be a set situated in the complex plane; α, ε positive real numbers, and λ_0 a complex number. A transformation H of the set S will be called an (α, ε)-**compression** from the point λ_0 (toward the origin) if this transformation is given by the formulas:

$$H(\lambda) = \begin{cases} \lambda - \lambda_0, & \text{if } |\lambda - \lambda_0| \leq \alpha, \\ \lambda, & \text{if } |\lambda - \lambda_0| \geq \alpha + \varepsilon, \\ \lambda - \lambda_0 h(\lambda), & \text{if } \alpha < |\lambda - \lambda_0| < \alpha + \varepsilon, \end{cases}$$

where

$$|\operatorname{Im} h(\lambda)| \leq \varepsilon, \qquad -\varepsilon \leq \operatorname{Re} h(\lambda) \leq 1 + \varepsilon.$$

Lemma 3. Let S be a set lying in the circle $|\lambda| \leq R$, $R > 0$. Then there exist an integer ν and a number $0 < K_1 < 1$, independent of R, complex numbers $\lambda_1, \dots, \lambda_\nu$, $|\lambda_j| = R$, and positive numbers $\alpha = \alpha(R)$ and $\varepsilon_1 = \varepsilon_1(R)$ such that the successive application of any ν (α, ε)-compactions, $0 \leq \varepsilon \leq \varepsilon_1$, with centers at the points $\lambda_1, \dots, \lambda_\nu$, transforms the set S into a set lying entirely in the circle with center at the origin and radius $R_1 \leq K_1 R$.

We note that it is enough to prove the lemma for the case when the set S coincides with the whole circle. From considerations of similarity it is clear that it suffices to restrict oneself to the unit circle, and the lemma becomes obvious.

Now we shall easily prove the theorem. Let $f \in C(\mathfrak{M})$,

$$\sup_{M \in \mathfrak{M}} |f(M)| = R,$$

so that the spectrum S of the function $f(M)$ is entirely situated in the circle of radius R with center at the origin. Choose numbers $\lambda_1, \dots, \lambda_\nu, \alpha, \varepsilon_1$ so that the assertion of Lemma 3 is fulfilled. Using Lemma 1, we successively construct functions $h_1, \dots, h_\nu \in B$ satisfying conditions 1), 3), and 4) of that lemma with $0 < \varepsilon' \leq \varepsilon_1$, and such that

$$h_1(M) = \begin{cases} 1, & \text{if } |f(M) - \lambda_1| \leq \alpha, \\ 0, & \text{if } |f(M) - \lambda_1| \geq \alpha + \varepsilon_1, \end{cases}$$

and, for $1 < k \leq \nu$,

$$h_k(M) = \begin{cases} 1, & \text{if } \left| f(M) - \sum_{j=1}^{k-1} \lambda_{jh} j(M) - \lambda_k \right| \leq \alpha, \\ 0, & \text{if } \left| f(M) - \sum_{j=1}^{k-1} \lambda_{jh} j(M) - \lambda_k \right| \geq \alpha + \varepsilon_1. \end{cases}$$

Let

$$f_1 = \sum_{j=1}^{\nu} \lambda_{jh} j.$$

Then $f_1 \in B$ and

$$\|f_1\| \leq \nu K \sup_{M \in \mathfrak{M}} |f(M)|. \tag{1}$$

At the same time, the spectrum of the function $f - f_1$, as is easy to see, is the result of applying to the spectrum of the function f ν successive (α, ε)-compactions, $0 \leq \varepsilon \leq \varepsilon_1$, with centers at the points $\lambda_1, \dots, \lambda_\nu$, and therefore, by Lemma 3,

$$\sup_{M \in \mathfrak{M}} |f(M) - f_1(M)| \leq K_1 \sup_{M \in \mathfrak{M}} |f(M)|. \tag{2}$$

In view of inequalities (1) and (2) and the arbitrariness of the choice of $f \in C(\mathfrak{M})$, to complete the proof it remains to apply Lemma 2.

The theorem just proved allows one to extend to nonsymmetric rings another result of Katznelson, also cited in (1). Let us first recall a definition formulated in (6). A function $\omega(z)$, defined on a certain set Ω of the complex plane, is, by definition, **acting** (operating) in the ring B if $\omega(f(M)) \in B$ for every function $f(M) \in B$ whose set of values (spectrum) belongs to Ω.

In work (1) Katznelson proved that if a semisimple ring B is regular and symmetric and in it there acts some* real-valued function $\omega(x)$, defined on the interval $-1 < x < 1$, such that

$$\omega(0) = 0, \qquad \text{but} \qquad \lim_{x \to 0} \frac{\omega(x)}{x} = \infty, \tag{3}$$

then $B = C(\mathfrak{M})$.

Katznelson' s proof is based on the "symmetric" version of Theorem 1.

* The continuity of $\omega(x)$ need not be assumed, since, as shown in (6), in nontrivial cases, when the ring B is infinite-dimensional, every function $\omega(x)$ acting in it must be continuous.

We shall say that a function $\omega(z)$ **acts in the broad sense** in the ring B if it acts in each of its quotient rings B_F. This definition applies to regular rings, and only in this case shall we use it.

Using this definition, one can reformulate Theorem 1 of the paper (^{5}) as follows: let Ω be a connected set on the real axis containing, besides zero, at least one more point, and let the real function $\omega(x)$ be defined on Ω, satisfy conditions (3), and act in the broad sense in a regular semisimple ring B. Then the idempotents are bounded in every quotient ring B_F.

Relying on this fact and on Theorem 1 proved above, we arrive at the following theorem:

Theorem 2. *Let B be a regular semisimple normed ring and let $\omega(x)$ be a real function defined on a connected subset of the real axis containing, besides zero, at least one more point. Suppose that $\omega(0) = 0$,*

$$\lim_{x \to 0} \frac{\omega(x)}{x} = \infty.$$

If $\omega(x)$ acts in the broad sense in the ring B, then $B = C(\mathfrak{M})$.

We note that Theorem 2 generalizes the corresponding result of Katznelson, since in the case of symmetric rings a real function $\omega(x)$, defined on the interval $-1 < x < 1$, which acts in the ring in his sense, also acts in this ring in the broad sense. In other cases this is, generally speaking, false. For example, in every antisymmetric ring, obviously, every real function acts.

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Note: Figure translations are in progress. See original paper for figures.