THE FIRST BOUNDARY-VALUE PROBLEM FOR A CERTAIN GENERAL LINEAR EQUATION

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Abstract

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THE FIRST BOUNDARY-VALUE PROBLEM FOR A CERTAIN GENERAL LINEAR EQUATION

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This article* is based on our preceding article ([1]), and we shall adhere to the notation introduced there.

Let $\mathcal{E}$ be a bounded convex (see ([1])) set of nonnegative integer vectors $\mathbf{k} = (k_1, \dots, k_n)$, containing some vector $\mathbf{k}^0 = (k_1^0, \dots, k_n^0)$, $k_j^0 > 0$, $j = 1, \dots, n$, and such that together with $\mathbf{k}$ the set $\mathcal{E}$ contains the projection of $\mathbf{k}$ onto the coordinate subspace of any number of dimensions; let also $\mathbf{k}^1, \dots, \mathbf{k}^N$ be vectors such that for any $\mathbf{k} \in \mathcal{E}$ the representation

$$\mathbf{k} = \sum_1^N \lambda_s \mathbf{k}^s \left(\lambda_s \geq 0, \ \sum_1^N \lambda_s \leq 1 \right).$$

holds. Further, let

$$\sum_{k,\mathbf{l} \in \mathcal{E}} \alpha_{k,\mathbf{l}}(\bar{x}) \xi_k \xi_l \geq \varkappa \sum_1^N (\xi_{k^s})^2, \tag{1}$$

$$\alpha_{k,\mathbf{l}}(\bar{x}) = \alpha_{l,\mathbf{k}}(\bar{x}), \qquad |\alpha_{k\mathbf{l}}(\bar{x})| \leq M,$$

$$E_G(f, \varphi) = \int_G \sum_{k,\mathbf{l} \in \mathcal{E}} \alpha_{k\mathbf{l}}(\bar{x}) f^{(\mathbf{k})}(\bar{x}) f^{(\mathbf{l})}(\bar{x}) \, dg,$$

$$E_G(f) = E_G(f, f), \qquad D_G(f) = \int_G \sum_1^N \left(f^{(\mathbf{k}^s)} \right)^2 \, dG,$$

where $\varkappa$ and M do not depend on $\bar{x} \in G \subset R_n$, and G is a regular bounded domain (see ([1])). Specify a function Φ with $E_G(\Phi) < \infty$.

The vectors $\mathbf{0} = \mathbf{l}^0 < \mathbf{l}^1 < \dots < \mathbf{l}^\nu < \mathbf{k}$, each of which exceeds the preceding one by one unit (one component is larger by one unit, and the others are equal), form a chain of the vector $\mathbf{k}$. To each $\mathbf{k}^s$ $(s = 1, \dots, N)$ let us assign one of its chains, and call the skeleton $\Lambda_{\mathcal{E}}$ the set of all vectors $\mathbf{l}$ entering into all these chains. In ([1]) it was established that the skeleton $\Lambda_{\mathcal{E}}$ determines the set of boundary functions for the function Φ: if $\mathbf{l} < \mathbf{l}' \in \Lambda_{\mathcal{E}}$ and $l_i < l_i^1$, then the derivative $\Phi^{(\mathbf{l})}$ has i-limit values $\gamma_{l,i}$, $\gamma \in L_2(\gamma_i)$ ($\gamma_i = \operatorname{pr}_i \gamma$ is the projection of γ onto $x_i = 0$) on a piece γ of the boundary Γ of the domain G in a sufficiently small neighborhood of a regular point $\bar{x}_0 \in \Gamma$.

Introduce the class $\mathfrak{M} = \mathfrak{M}(G; \Phi)$ of functions f with $E_G(f) < \infty$, having the same set of boundary functions as Φ. One may also say that the functions $f \in \mathfrak{M}$ are those functions with $E_G(f) < \infty$ for which $f - \Phi \in \mathfrak{M}_0$, i.e. $f - \Phi$

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* The main ideas of this article in simpler, but characteristic, cases were reported at the seminar on function theory of the V. A. Steklov Mathematical Institute of the Academy of Sciences of the USSR in February 1961.

has a set of boundary functions identically equal to zero on Γ. Let us note that the functions of the classes $\mathfrak{M}$ and $\mathfrak{M}_0$ are described completely by the means of ordinary mathematical analysis (see (1)). Here we do not deal with classes of functions that we cannot describe by these means.

Theorem 1. *There exists, moreover, a unique function $u \in \mathfrak{M}$ for which*

$$\min_{f \in \mathfrak{M}} E_G(f) = E_G(u).$$

The proof is based on an inequality (of Poincaré type)

$$\left\|f^{(\mathbf{k})}\right\|_{L_2(G)} \leq c D_G(f), \qquad \mathbf{k} \in \mathfrak{E} \quad \text{for all } f \in \mathfrak{M}_0.$$

Then, for a minimizing sequence $\{f_p\}$,

$$\left\|f_p^{(\mathbf{k})} - f_q^{(\mathbf{k})}\right\|_{L_2(G)} \leq c_1 D_G(f_p - f_q) \leq c_2 E_G(f_p - f_q) \to 0, \tag{2}$$

$$\mathbf{k} \in \mathfrak{E}, \qquad p, q \to \infty.$$

By virtue of (12) from (1), $\left\|f_p^{(\mathbf{k})}\right\|_{L_2(G')} < \infty$ $(p = 1, 2, \dots)$ for all $G' \in \overline{G'} \subset G$. This is sufficient, taking into account (2) and the nonnegativity of the integrand in $E_G(f)$, to conclude the existence of a function u, for which $\left\|u^{(\mathbf{k})}\right\|_{L_2(G')} < \infty$ for $\mathbf{k} \in \mathfrak{E}$ and all G', and that $E_G(f_p - u) \to 0$ $(p \to \infty)$. The fact that u has the same set of boundary functions as f_p $(p = 1, 2, \dots)$, or Φ, is proved with the aid of (2) and inequality (14) from (1), in the left-hand side of which, as $\mu_{k,\gamma,i}$, one must take the corresponding boundary function of the function $u - \Phi$, or, what is the same, $u - f_p$, and on the right-hand side replace Φ by $u - \Phi$.

As usual, u may also be characterized as a function of the class $\mathfrak{M}$ for which the equation (in variations)

$$E_G(u, v) = 0 \qquad \text{for all } v \in \mathfrak{M}_0, \tag{3}$$

is satisfied; and if the coefficients $\alpha_{\mathbf{kl}}(\bar{x})$ have continuous partial derivatives in G up to orders $\mathbf{l}$ inclusive, and u is continuous together with its partial derivatives $u^{(\mathbf{k}+\mathbf{l})}$, then u satisfies in G the differential equation

$$Lu = \sum_{k,\mathbf{l} \in \mathfrak{E}} (-1)^{|\mathbf{l}|} D^{(\mathbf{l})}(\alpha_{\mathbf{kl}} u) = 0, \qquad |\mathbf{l}| = \sum_1^n l_j. \tag{4}$$

In the case where $\alpha_{\mathbf{kl}} = \alpha_{\mathbf{kl}}$ are constant coefficients, condition (1) may be replaced by the more general condition

$$\theta(\xi) = \sum_{k,\mathbf{l} \in \mathfrak{E}} \alpha_{\mathbf{kl}} i^{|\mathbf{k}|-|\mathbf{l}|} \xi^{k+\mathbf{l}} \geq \chi \sum_{s=1}^{N} (\xi^{k_s})^2, \qquad \xi^k = \xi_1^{k_1} \cdots \xi_n^{k_n}, \tag{5}$$

$$\mathbf{k} = (k_1, \dots, k_n),$$

and Theorem 1 will be true. This was proved under the assumption that the function Φ with $E_G(\Phi)$ can be extended to some domain $G_1 \supset \overline{G}$ in such a way that for the extended function $\overline{\Phi}$ one also has $E_G(\overline{\Phi}) < \infty$.

Let $\overline{\mathfrak{E}}$ be the smallest (continuous) convex body stretched over $\mathfrak{E}$, and let $\overline{\mathfrak{E}}_i$ be its projection onto the plane $x_i = 0$. The boundary ν_i of the body $\overline{\mathfrak{E}}_i$ consists of two parts: $\nu_i = \nu_i^{(k)} + \nu_i^{(b)}$, where $\nu_i^{(k)}$ is the part of ν_i lying in the coordinate planes $x_j = 0$ $(j = 1, \dots, i-1, i+1, \dots, n)$, and $\nu_i^{(b)}$ is the closure of the remaining part, which we shall call the lateral boundary of $\overline{\mathfrak{E}}_i$. If $\mathbf{k} = (k_1, \dots, k_n) \in \mathfrak{E}$, then put $\mathbf{k}^{e_i} = (k_1, \dots, k_{i-1}, 0, k_{i+1}, \dots, k_n)$.

We shall assume that α_{k1} are constants (not depending on $\bar{x}$) and require that the following restriction be satisfied:

Condition A. If $\alpha_{k1} \neq 0$ and $k_i + l_i > 0$, then one of the vectors k^{e_i}, l^{e_i} (more precisely, its endpoint) must not belong to v_i. It can be shown that under these restrictions the polynomial $Q(\xi)$ satisfies the conditions

$$\lim_{\left(\sum_1^n \xi_j^2\right)^{1/2} \to \infty} \frac{\partial Q / \partial \xi_j}{Q} = 0, \qquad (j = 1, \dots, n)$$

for real ξ_j, and this is Hörmander' s criterion ([2]) for the infinite differentiability of the generalized solution u of equation (4), i.e., of the function u for which (2) holds.

Thus (provided (1) holds), condition A is sufficient for the infinite differentiability of the generalized solution of equation (1). One could obtain a necessary and sufficient condition, but it is cumbersome.

From (1) follows (5) for any $\bar{x} \in G$, and therefore, if $\alpha_{k1}(\bar{x})$ are infinitely differentiable, then on the basis of the corresponding Hörmander criterion ([3]) the generalized solution (4) with variable coefficients is infinitely differentiable, i.e., (4) is an equation of hypoelliptic type.

Let us note that the equation

$$(-1)^{r_1} \frac{\partial^{2r_1} u}{\partial x_1^{2r_1}} + (-1)^{r_2} \frac{\partial^{2r_2} u}{\partial x_2^{2r_2}} + (-1)^{r_1+r_2} \frac{\partial^{2r_1+2r_2} u}{\partial x_1^{2r_1} \partial x_2^{2r_2}} = 0 \tag{6}$$

is not hypoelliptic, and at the same time the theory set forth above is valid for it; thus, for it the boundary-value problem described above (of the first kind) is posed correctly: in the class generated by the functional $E_G(\Phi)$ corresponding to (6), there exists a unique generalized solution of (6).

The question of uniqueness of the classical solution (4) will be the subject of another note.

In conclusion I wish to mention the work of K. Zh. Nauryzbaev ([4]), where, for one equation of hypoelliptic type with constant coefficients of type (4), a boundary-value problem in an infinite half-space is solved by the variational method. Let us also mention works in which boundary-value problems for equations of hypoelliptic type are solved by other methods: M. Nicolesco ([5]), Pini Bruno ([6]), P. P. Mosolov ([7]).

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Note: Figure translations are in progress. See original paper for figures.