



---

Soviet-era science, translated into English

# Doklady of the Academy of Sciences of the USSR

1962

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196201.87395>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

## Abstract

## Full Text

Doklady of the Academy of Sciences of the USSR  
1962. Volume 147, No. 6

## MATHEMATICS

M. ROSENBLATT-ROT

# ON THE LAW OF LARGE NUMBERS AND ON THE STRONG LAW OF LARGE NUMBERS FOR NONHOMOGENEOUS MARKOV CHAINS

(Presented by Academician A. N. Kolmogorov on June 27, 1962)

Let  $\alpha_i$  be the coefficient of ergodicity of the  $i$ -th probability transition function of a certain Markov chain, nonhomogeneous in general (see <sup>(1,2)</sup>), and denote

$$\alpha^{(n)} = \min_{1 \leq i \leq n} \alpha_i \quad (n \in I; I = (1, 2, \dots)).$$

If  $l$  is some natural number ( $l > 1$ ), let

$$\sigma_m^{-1} = l^{2m} \alpha(l^{m+1}), \quad \omega_i^{-2} = \sum_{m=\rho}^{\infty} \sigma_m,$$

where  $\rho = [\log i / \log l]$ .

**Theorem 1.** In order that a sequence of random variables  $\xi_i$  ( $i \in I$ ), connected into a nonhomogeneous Markov chain for which  $\alpha^{(n)} > 0$  ( $n \in I$ ),  $n\alpha^{(n)} \rightarrow \infty$  ( $n \rightarrow \infty$ ), and depending on  $r \geq 1$  moments of time, obey the law of large numbers, it is sufficient that

$$\lim_{n \rightarrow \infty} \frac{1}{n^2 \alpha^{(n)}} \sum_{i=1}^n \mathbf{D}\xi_i = 0.$$

**Theorem 2.** If a sequence of random variables  $\xi_i$  ( $i \in I$ ), connected into a nonhomogeneous Markov chain with  $\alpha^{(n)} > 0$  ( $n \in I$ ),  $n\alpha^{(n)} \rightarrow \infty$  ( $n \rightarrow \infty$ ), satisfies the condition

$$\sum_{n=1}^{\infty} \frac{\mathbf{D}\xi_n}{\omega_n^2} < \infty,$$

then it obeys the strong law of large numbers.

If

$$\sigma_{m+1} < k\sigma_m \quad (k < 1; m \geq \rho),$$

this condition takes the form

$$\sum_{n=1}^{\infty} \frac{\mathbf{D}\xi_n}{n^2\alpha^{(n)}} < \infty.$$

These results are obtained with the aid of several lemmas.

**Lemma 1.** If the random variables  $\xi_i$  ( $i \in I$ ) are connected into a nonhomogeneous Markov chain with  $\alpha^{(n)} > 0$  ( $n \in I$ ) and depend on  $r \geq 1$  moments of time, then

$$\mathbf{D} \left( \sum_{k=1}^n \xi_k \right) \leq \frac{C}{\alpha^{(n)}} \sum_{k=1}^n \mathbf{D}\xi_k.$$

**Lemma 2.** If the random variables  $\xi_i$  ( $i \in I$ ) are connected into a nonhomogeneous Markov chain with  $\alpha^{(n)} > 0$  ( $n \in I$ ), then for any  $\varepsilon > 0$  the following holds:

inequality

$$\mathbf{P} \left\{ \max_{1 \leq s \leq n} \left| \sum_{k=1}^s (\xi_k - \mathbf{M}\xi_k) \right| > \varepsilon \right\} \leq \frac{1}{\varepsilon_1^2 \alpha^{(n)}} \sum_{i=1}^n \mathbf{D}\xi_i,$$

where  $\varepsilon_1$  is a certain positive number depending on  $\varepsilon$  and on the chain.

**Lemma 3.** Let the random variables  $\xi_i$  ( $i \in I$ ) be associated with an inhomogeneous Markov chain with  $\alpha^{(n)} > 0$  ( $n \in I$ ); if  $c_i > 0$ ,  $c_i \geq c_{i+1}$  ( $i \in I$ ),  $1 \leq n \leq m$ ,  $\varepsilon > 0$ , then the inequality

$$\mathbf{P} \left\{ \max_{n \leq k \leq m} c_k \left| \sum_{s=1}^k (\xi_s - \mathbf{M}\xi_s) \right| \geq \varepsilon \right\} \leq \frac{1}{\varepsilon_1^2 \alpha^{(m)}} \left\{ c_n^2 \sum_{k=1}^n \mathbf{D}\xi_k + \sum_{k=n+1}^m c_k^2 \mathbf{D}\xi_k \right\},$$

where  $\varepsilon_1$  is a certain positive number depending on  $\varepsilon$  and on the chain.

For  $\alpha^{(n)} = O(n^{-\beta})$  ( $0 \leq \beta < 1$ ), Theorem 1 yields Theorem 1 of <sup>(3)</sup>; from Theorem 2, if one takes  $k = l^{2-2\beta} < 1$ , one obtains Theorem 1 of <sup>(4)</sup>; from Lemma 2—Lemma 1 of <sup>(4)</sup>, and from Lemma 3—Theorem 1 of <sup>(5)</sup>.

We note that for  $\alpha^{(n)} \equiv 1$  ( $n \in I$ ), i.e. for independent random variables, Theorem 1 implies the result of A. A. Markov, while Theorem 2 and Lemma 2 give the results of A. N. Kolmogorov <sup>(6-9)</sup>, and Lemma 3 gives the result of Hájek—Rényi <sup>(10)</sup>.

In view of the fact that, for arbitrarily dependent random variables, the inequality

$$\mathbf{D} \left( \sum_{k=1}^n \xi_k \right) \leq n \sum_{k=1}^n \mathbf{D} \xi_k$$

holds, the results contained in Theorems 1 and 2 are of interest only when  $n\alpha^{(n)} \rightarrow \infty$  ( $n \rightarrow \infty$ ), because otherwise, applying this inequality instead of the inequality of Lemma 1, we obtain better results. Under these conditions, if one assumes that  $n\alpha^{(n)}$  is a monotone sequence, we obtain that  $\sigma_m$  is a nonincreasing sequence, so that the requirement imposed on it in Theorem 2 is not a very strong restriction. We note that this requirement is satisfied for  $\alpha^{(n)} = (\log n)^{-\beta}$  ( $\beta > 0$ );  $\alpha^{(n)} = (\log_p n)^{-\beta}$  ( $p > 1$ ) ( $\log_p n = \log(\log_{p-1} n)$ ,  $0 < \beta \leq 1$ );  $\alpha^{(n)} = n^{-\beta}(\log n)^{-\gamma}$  ( $\beta + \gamma < 2$ ,  $\beta > 0$ ,  $\gamma > 0$ );  $\alpha^{(n)} = n^{-\beta}$  ( $0 \leq \beta < 1$ ).

Faculty of Mathematics and Physics

K. I. Parhon University

Bucharest, Romania

Received

24 III 1962

## CITED LITERATURE

<sup>1</sup> R. L. Dobrushin, *Theory of Probability and Its Applications*, **1**, issue 1, 72 (1956).

<sup>2</sup> R. L. Dobrushin, *Theory of Probability and Its Applications*, **1**, issue 4, 365 (1956).

<sup>3</sup> M. Rosenblatt-Roth, DAN, **134**, No. 2, 278 (1960).

<sup>4</sup> M. Rosenblatt-Roth, DAN, **141**, No. 6, 1310 (1961).

<sup>5</sup> M. Rosenblatt-Roth, C. R., in press.

<sup>6</sup> A. Kolmogoroff, Math. Ann., **99**, 309 (1928).

<sup>7</sup> A. Kolmogoroff, Math. Ann., **102**, 484 (1929).

<sup>8</sup> A. Kolmogoroff, C. R., **191**, 910 (1930).

<sup>9</sup> A. N. Kolmogorov, *Basic Concepts of Probability Theory*, Moscow—Leningrad, 1936.

<sup>10</sup> J. Hájek, A. Rényi, Acta Math. Acad. Sci. Hung., **6**, No. 3–4, 281 (1955).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*