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Abstract

Full Text

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ON A CLASS OF LOCALLY CONVEX SPACES

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In the present note a new class of locally convex spaces is introduced, including most of the functional spaces used in analysis. To these spaces (we shall call them α-spaces) the Banach theorems on the open mapping and on the closed graph are generalized. The note also studies the connection of α-spaces with complete spaces and gives necessary and sufficient conditions for the completeness and hypercompleteness of a separable Montel space and of the space conjugate to an LF-space.

Definition 1. We shall call a locally convex space an α-**space** if, in its weak dual, every sequentially closed subspace is closed.*

It follows from this definition that if X is an α-space, then it is an α-space in all topologies giving the same dual.

Theorem 1. *If in a locally convex space X every sequentially closed subspace is closed, then its dual X' is an α-space in all topologies t,*

$$\sigma(X', X) \leq t \leq \tau(X', X),$$

*where $\sigma(X', X)$ is the weak topology and $\tau(X', X)$ is the Mackey topology.**
*

Corollary 1. *The dual X' of a metrizable locally convex space X is an α-space in the topology t,*

$$\sigma(X', X) \leq t \leq \tau(X', X).$$

In particular, a semireflexive DF-space (see [2]) is an α-space.

Corollary 2. *A space of type M^* [3,4] is an α-space.*

Other examples of α-spaces will be given below.

Theorem 2. *Closed subspaces and quotient spaces by closed subspaces of α-spaces are α-spaces.*

Definition 2. We shall call a locally convex space an α'-**space** if, in its Mackey topology, its weak dual is sequentially complete.***

Theorem 3. *A linear mapping with closed graph from an α-space onto an α'-space is open. In particular, a continuous linear mapping from an α-space onto an α'-space is open.*

Theorem 4. *A linear mapping with closed graph from an α'-space into an α-space is continuous.*

Theorem 5. *The following assertions about an α'-space X are equivalent:*

1) X is an α-space;
2) every continuous linear mapping of the space X onto an α'-space is open;
3) every linear mapping with closed graph of the space X onto an α'-space is open;

* A set is called sequentially closed if it contains the limits of its convergent sequences.
** Concerning the topologies $\sigma(X', X)$ and $\tau(X', X)$, see ([1]).
*** Such, in particular, are all barrelled spaces.

4) every linear mapping with closed graph from an α'-space into the quotient space of the space X by a closed subspace is continuous;
5) every linear mapping with closed graph from an α'-space onto the quotient space of the space X by a closed subspace is continuous.

Theorem 6. If X is a barrelled α-space, then (X, t) is complete* for

$$t_N \leq t \leq \tau(X, X'),$$

where t_N is the topology generated by the polars of $\sigma(X', X)$-convergent sequences of the space X'**.

Theorem 7. A separable complete space is an α-space.

From Theorems 6 and 7 it follows:

Corollary. For a separable barrelled space, completeness coincides with membership in the class of α-spaces. In particular, a separable F-space is an α-space.

There exist Banach spaces which are not α-spaces. Thus, if X is a nonreflexive Banach space, sequentially complete in the weak topology $\sigma(X, X')$***, then its strong dual is not an α-space. Such are, for example, the spaces m of bounded numerical sequences and M of bounded measurable functions on the interval $(0, 1)$.

Theorem 8. In order that a Montel space (see ([1])) be an α-space, it is necessary and sufficient that in its strong dual every sequentially closed subspace be closed.

Theorem 9. In order that a separable Montel space be complete (hypercomplete****), it is necessary and sufficient that in its strong dual every sequentially closed subspace (closed convex circled set) be closed.

By virtue of this theorem, in order to establish the completeness of the spaces D or D' of L. Schwartz ([10])*****, it is enough to show that in their strong duals (respectively D' and D) every sequentially closed subspace is closed.

Theorem 10. Let X be an LF-space ([12]), (X', t) the dual space in a topology t,

$$t_N \leq t \leq \tau(X', X),$$

where t_N is the topology generated by the polars of convergent sequences of the space X. Then, for the completeness (hypercompleteness) of the space (X', t) it is necessary and sufficient that in X every sequentially closed subspace (closed convex circled set) be closed******.

Remark. For a reflexive LF-space and $t = \tau(X', X)$ this theorem was proved in ([13]).

Remark added in proof.

Definition 1′. We shall call a locally convex space a γ-space if in its weak dual every subspace containing the limit points of its bounded subsets is closed. Complete spaces and α-spaces are γ-spaces. Closed subspaces and quotient spaces by them of γ-spaces are γ-spaces.

Definition 2′. We shall call a locally convex space in the Mackey topology a γ'-space if its weak dual is quasicomplete. Barrelled spaces are γ'-spaces.

* On complete spaces see ([5–7]).

** From a result of D. A. Raikov ([9]) it follows that (X, t) remains complete also for many weaker topologies t.

*** Examples of such spaces are the spaces l and $L(0, 1)$ (see Chapter IX, § 4 in ([9])).

**** On hypercomplete spaces see ([7]).

***** The spaces K and K' in the notation adopted in ([11]).

****** From a result of D. A. Raikov ([9]) there follows the validity of this theorem also for many weaker topologies t.

Theorems 3, 4, and 5 remain valid if, in their formulations, α-spaces are replaced by γ-spaces, and α'-spaces by γ'-spaces.

Theorem 6′. *A barrelled γ-space is complete.*

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Note: Figure translations are in progress. See original paper for figures.