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Abstract

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A NOTE ON COMMUTATIVE SEMIGROUPS

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1. If c is a two-sided ideal of the semigroup $\mathfrak{B}$, $\mathfrak{D} = \mathfrak{B} \setminus c$, then, following ([1]), we shall assume that $\mathfrak{B}/c = \mathfrak{D} = \mathfrak{D} \cup \bar{0}$, where $\bar{0}$ is the zero of $\mathfrak{B}/c$. We shall call $\mathfrak{B}$ an extension of c by means of $\mathfrak{D}$.

2. By $\mathfrak{K}$ we denote the class of commutative semigroups with zero, containing a finite number of nonregular elements (([1]), p. 104). It can be shown that the class $\mathfrak{K}$ consists of those and only those commutative semigroups with zero which have a finite ideal composition series (in the sense of Rees ([2])). In particular, the class $\mathfrak{K}$ contains all finite commutative semigroups with zero. We note that the requirement that a zero be present in the semigroups of the class $\mathfrak{K}$ is inessential and is adopted only to simplify the formulations. A commutative semigroup $\mathfrak{A}$ will be called elementary if

$$\mathfrak{A} = \mathfrak{G} \cup \mathfrak{N}, \quad \mathfrak{G} \cap \mathfrak{N} = \emptyset^*,$$

where $\mathfrak{G}$ is a group with identity e, $\mathfrak{N}$ is a nilpotent finite ideal of $\mathfrak{A}$, and e is the identity of $\mathfrak{A}$. Elementary semigroups belong to $\mathfrak{K}$; they are described without difficulty: their description is readily reduced to the description of finite nilpotent semigroups, and the latter are described in ([3]). An ideal series of a commutative semigroup $\mathfrak{A}$ will be called reduced if all its factors are elementary semigroups.

3. **Theorem 1.** Let $\mathfrak{A} \in \mathfrak{K}$, $\mathfrak{A}^2 = \mathfrak{A}$. Then:

α) $\mathfrak{A}$ has a reduced series;

β) all terms of any reduced series of $\mathfrak{A}$ are ideals of $\mathfrak{A}$;

γ) if

$$\begin{gathered} \mathfrak{A} = \mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \cdots \supset \mathfrak{A}_n \supset \mathfrak{A}_{n+1} = 0, \\ \mathfrak{A} = \mathfrak{B}_1 \supset \mathfrak{B}_2 \supset \cdots \supset \mathfrak{B}_m \supset \mathfrak{B}_{m+1} = 0 \end{gathered} \tag{1}$$

are reduced series of $\mathfrak{A}$, then $m = n$, and between the factors of these series one can establish a one-to-one correspondence ρ such that, if $\mathfrak{A}_i/\mathfrak{A}_{i+1}$ and $\mathfrak{B}_j/\mathfrak{B}_{j+1}$ correspond under ρ, then

$$\mathfrak{A}_i \setminus \mathfrak{A}_{i+1} = \mathfrak{B}_j \setminus \mathfrak{B}_{j+1}.$$

From γ) it follows, in particular, that any two reduced series of $\mathfrak{A} \in \mathfrak{K}$, $\mathfrak{A}^2 = \mathfrak{A}$, are isomorphic.

4. Let $\mathfrak{A} \in \mathfrak{K}$, $\mathfrak{A}^2 = \mathfrak{A}$, and let (1) be a reduced series of $\mathfrak{A}$, $p_i = \mathfrak{A}_i \setminus \mathfrak{A}_{i+1}$, $i \leq n$. Since $\mathfrak{A}_i/\mathfrak{A}_{i+1}$ is an elementary semigroup (by the definition of a reduced series), $\mathfrak{A}_i/\mathfrak{A}_{i+1}$ contains an identity. Hence there exists $e_i \in p_i$ such that e_i is an identity for the elements of p_i.

Denote by a_k the ideal homomorphism of $\mathfrak{A}$ generated by $\mathfrak{A}_k$, a term of (1) (a_k has meaning in view of Theorem 1 β). Put, for $x \in p_i$, $i \leq k \leq n$:

$$\varphi_{ki} x = a_{k+1}(x e_k).$$

φ_{ki} maps p_i into $\bar{p}_k = \mathfrak{A}_k/\mathfrak{A}_{k+1}$. We collect the mappings φ_{ki} into the matrix

$$\Phi = \begin{pmatrix} \varphi_{11} & 0 & \cdots & 0 & & \\ \varphi_{21} & \varphi_{22} & \cdots & 0 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nn} & & \end{pmatrix}.$$

* By $\emptyset$ the empty set is denoted.

Consider the collection $V(\mathfrak{A})$ of all n-dimensional vectors X_i of the form $X_i = (0, \dots, 0, x_i, \varphi_{i+1,i} x_i, \dots, \varphi_{ni} x_i)$ (the zeros stand in the first $i-1$ places of X_i), $x_i \in \mathfrak{p}_i$. The zero standing in the k-th place of X_i is regarded as the zero of $\bar{\mathfrak{p}}_k$. If $X \in V(\mathfrak{A})$, then by $\{X\}_k$ we denote the k-th component of X. From the definition of X_i and φ_{ki} it follows that $\{X\}_k \in \bar{\mathfrak{p}}_k$ for all $X \in V(\mathfrak{A})$ and $k = 1, 2, \dots, n$. Therefore one may define multiplication of vectors $X, Y \in V(\mathfrak{A})$:

$$\{X \cdot Y\}_k = \{X\}_k \cdot \{Y\}_k. \tag{2}$$

Theorem 2. With respect to multiplication (2), $V(\mathfrak{A})$ is a semigroup isomorphic to $\mathfrak{A}$.

Thus every semigroup $\mathfrak{A} \in \mathfrak{K}$, $\mathfrak{A}^2 = \mathfrak{A}$, can be represented isomorphically by a semigroup of vectors whose components belong to elementary semigroups. At the same time, Theorem 2 implies:

Theorem 3. The specification of the factors $\bar{\mathfrak{p}}_i = \mathfrak{A}_i/\mathfrak{A}_{i+1}$ and of the matrix Φ completely determines the semigroup $\mathfrak{A}$.

Theorem 2 also leads to the proposition:

Theorem 4. If an arbitrary semigroup $\mathfrak{A}$ has a finite ideal series all of whose factors are elementary semigroups, then $\mathfrak{A}$ is commutative and $\mathfrak{A} \in \mathfrak{K}$.

In view of Theorem 3 the notation has meaning: $\mathfrak{A} = (\bar{\mathfrak{p}}_1, \dots, \bar{\mathfrak{p}}_n, \Phi)$.

5. Let $\mathfrak{A}, \mathfrak{B} \in \mathfrak{K}$, $\mathfrak{A}^2 = \mathfrak{A}$, $\mathfrak{B}^2 = \mathfrak{B}$,
   $\mathfrak{A} = (\overline{\mathfrak{p}}_1, \dots, \overline{\mathfrak{p}}_n, \Phi)$,
   $\mathfrak{B} = (\overline{\mathfrak{q}}_1, \dots, \overline{\mathfrak{q}}_m, \Psi)$.

Suppose that there is a non-identical substitution Λ of the set $1, 2, \dots, n$ such that for every $i = 1, 2, \dots, n$ there exists an isomorphism λ_i of $\overline{\mathfrak{p}}_i$ onto $\overline{\mathfrak{q}}_{\Lambda i}$. Denote here and in Theorem 5 by L the square matrix of degree n such that in the i-th column of L, in the row with number Λi, there stands λ_i, while the remaining elements of the i-th column of L are zeros. Naturally, L^{-1} is defined.

Theorem 5. Let $\mathfrak{A}, \mathfrak{B} \in \mathfrak{K}$, $\mathfrak{A}^2 = \mathfrak{A}$, $\mathfrak{B}^2 = \mathfrak{B}$,
$\mathfrak{A} = (\overline{\mathfrak{p}}_1, \dots, \overline{\mathfrak{p}}_n, \Phi)$,
$\mathfrak{B} = (\overline{\mathfrak{q}}_1, \dots, \overline{\mathfrak{q}}_m, \Psi)$. For $\mathfrak{A}$ and $\mathfrak{B}$ to be isomorphic it is necessary and sufficient that $m = n$ and that there be found a matrix L such that $\Psi = L\Phi L^{-1}$ (multiplication of the nonzero elements of L, Φ, L^{-1} is understood in the sense of composition of mappings).

6. We shall describe briefly how semigroups of the class $\mathfrak{K}$ can be constructed. It is enough to restrict ourselves to semigroups $\mathfrak{A} = \mathfrak{K}$, $\mathfrak{A}^2 = \mathfrak{A}$, since one can always adjoin an identity e to a semigroup $\mathfrak{A}$, and if $\mathfrak{A}' = \mathfrak{A} \cup e$ and $\mathfrak{A} \in \mathfrak{K}$, then $\mathfrak{A}' \in \mathfrak{K}$.

6.1. Let $\mathfrak{p}$ be a set with a partial binary operation (multiplication), i.e. an operation defined not on all of $\mathfrak{p} \times \mathfrak{p}$. If $x, y \in \mathfrak{p}$ and $x \cdot y$ is not defined in $\mathfrak{p}$, we shall write $x \cdot y \notin \mathfrak{p}$. Construct $\overline{\mathfrak{p}} = \mathfrak{p} \cup \overline{0}$, putting: $x \cdot y = \overline{0}$ if $x \cdot y \notin \mathfrak{p}$, and $x \cdot \overline{0} = \overline{0} \cdot x = \overline{0} \cdot \overline{0} = \overline{0}$, for $x, y \in \mathfrak{p}$. We shall call $\overline{0}$ the zero of $\overline{\mathfrak{p}}$. We shall say that $\mathfrak{p}$ is a semigroupoid if $\overline{\mathfrak{p}}$ is a semigroup.

6.2. Let $\mathfrak{p}$ be a semigroupoid, and let $\mathfrak{q}$ be a set with a partial single-valued binary operation—multiplication. A mapping $\varphi : \mathfrak{p} \to \mathfrak{q}$ will be called a homomorphism of $\mathfrak{p}$ if the implication holds: $x, y, x \cdot y \in \mathfrak{p} \to \varphi x \cdot \varphi y \in \mathfrak{q}$, $\varphi x \cdot \varphi y = \varphi(x \cdot y)$. If the homomorphism $\varphi : \mathfrak{p} \to \mathfrak{q}$ can be extended to a homomorphism $\overline{\varphi} : \overline{\mathfrak{p}} \to \overline{\mathfrak{q}}$, with $\varphi \overline{0}$ the zero of $\overline{\mathfrak{q}}$, then φ will be called an exact homomorphism of $\mathfrak{p}$. From the definition itself it follows that exact homomorphisms of $\mathfrak{p}$ are completely determined by homomorphisms of the semigroup $\overline{\mathfrak{p}}$.

If $\mathfrak{R}$ is a subset of $\mathfrak{q}$, then, putting for $x, y \in \mathfrak{R}$ $x \circ y = x \cdot y$ if $x \cdot y \in \mathfrak{R}$, we turn $\mathfrak{R}$ into a set with a partial operation. We shall say that the operation in $\mathfrak{R}$ is induced by the operation in $\mathfrak{q}$. If φ is an exact homomorphism of $\mathfrak{p}$ into $\mathfrak{q}$, then $\varphi\mathfrak{p}$ will be a semigroupoid with respect to the operation induced in $\varphi\mathfrak{p}$ by the operation in $\mathfrak{q}$. A homomorphism of $\mathfrak{p}$ that is a one-to-one mapping will be called a monomorphism of $\mathfrak{p}$.

Theorem 6. *Every homomorphism φ of a semigroupoid $\mathfrak{p}$ is representable in the form $\varphi = \mu\psi$, where ψ is an exact homomorphism of $\mathfrak{p}$, and μ is a monomorphism of the semigroupoid $\psi\mathfrak{p}$.*

This theorem reduces the question of describing homomorphisms of a semigroupoid $\mathfrak{p}$ into the given structure to the description of abstract homomorphisms (i.e. two-sided stable equivalences) of the semigroup $\bar{\mathfrak{p}}$ and to the description of monomorphisms of a certain semigroupoid $\psi\mathfrak{p}$ into the given structure.

6.3. Clifford ([4]), in somewhat different terms, proved:

Theorem. *Let $\mathfrak{c}$ be a semigroup with identity, $\mathfrak{D}$ a semigroupoid, and $\mathfrak{p}$ a homomorphism of $\mathfrak{D}$ into $\mathfrak{c}$. Define in $\mathfrak{B} = \mathfrak{c} \cup \mathfrak{D}$ the multiplication "$\circ$" :*

$$X, Y \in \mathfrak{c} \to X \circ Y = XY; \qquad X, Y, XY \in \mathfrak{D} \to X \circ Y = XY;$$

$$X \in \mathfrak{D}, \quad Y \in \mathfrak{c} \to X \circ Y = \varphi X \cdot Y, \qquad Y \circ X = Y \cdot \varphi X;$$

$$X, Y \in \mathfrak{D}, \quad XY \notin \mathfrak{D} \to X \circ Y = \varphi X \cdot \varphi Y.$$

Then with respect to the multiplication "$\circ$" $\mathfrak{B}$ is a semigroup that is an extension of $\mathfrak{c}$ by means of the semigroup $\overline{\mathfrak{D}}$.

Every extension of $\mathfrak{c}$ by means of $\overline{\mathfrak{D}}$ can be obtained in the manner described in the theorem.

6.4. The construction of semigroups $\mathfrak{A} \in \mathfrak{K}$, $\mathfrak{A}^2 = \mathfrak{A}$, which we propose consists in the successive application of the operation of extension of an elementary semigroup $\mathfrak{c}$ by means of a semigroup $\mathfrak{p} \in \mathfrak{K}$. From Theorems 1 and 4 it follows that in this way all semigroups $\mathfrak{A} \in \mathfrak{K}$, $\mathfrak{A}^2 = \mathfrak{A}$, and only they, will be obtained.

Since $\mathfrak{c}$ is an elementary semigroup, $\mathfrak{c}$ contains an identity. Therefore, by 6.3, the description of the indicated extension is reduced to the description of homomorphisms of $\mathfrak{p}$ into $\mathfrak{c}$. This, in turn, in view of Theorem 7, is reduced to the description of abstract homomorphisms of the semigroup $\bar{\mathfrak{p}}$ and monomorphisms of semigroupoids of the form $\psi\mathfrak{p}$ (ψ is an exact homomorphism of $\mathfrak{p}$) into $\mathfrak{c}$. The semigroup $\bar{\mathfrak{p}}$ should be regarded as known; then its homomorphisms can be found as indicated in ([4]). Let $\mathfrak{q} = \psi\mathfrak{p}$; it can be shown that $\bar{\mathfrak{q}} \in \mathfrak{K}$. As a result, everything is reduced to the description of monomorphisms of semigroupoids $\mathfrak{q}$, $\bar{\mathfrak{q}} \in \mathfrak{K}$, into elementary semigroups. This problem, in turn, can be reduced to the description of monomorphisms into elementary semigroups of semigroupoids $\mathfrak{R}$ such that $\overline{\mathfrak{R}}$ is a nilpotent semigroup. Such monomorphisms are described in Theorem 6.

6.5. We shall say that $\mathfrak{R}$ is a nilpotent semigroupoid of degree n, if $\overline{\mathfrak{R}}^n = 0$. Let $\mathfrak{A}$ be an arbitrary semigroup, $\Omega \subset \mathfrak{c}$ (in Ω the operation in $\mathfrak{A}$ is not taken into account). Denote by Ω_n the collection of all words in the alphabet Ω whose length does not exceed n. Then Ω_n is a nilpotent semigroupoid of degree n with respect to the usual multiplication of words.

Let $\bar{I} = I \cup \bar{0}$ be a two-sided ideal of $\overline{\Omega}_n$, $I \cap \Omega = \emptyset$ (the case $I = \emptyset$ is not excluded). Put $(\Omega, n, I) = \Omega_n \setminus I$; (Ω, n, I) is also a nilpotent semigroupoid of degree n with respect to multiplication of words. The embedding of Ω in $\mathfrak{A}$ induces a homomorphism $f : (\Omega, n, I) \to \mathfrak{A}$.

Theorem 7. *A one-to-one mapping ν of a nilpotent semigroupoid $\mathfrak{N}$ of degree n into a semigroup $\mathfrak{A}$ is a monomorphism of $\mathfrak{N}$ if and only if there exist a semigroupoid (Ω, n, I), $\Omega \subset \mathfrak{A}$, and an exact homomorphism φ of (Ω, n, I) onto $\mathfrak{N}$ such that $f = \nu\varphi$ (f is the homomorphism of (Ω, n, I) into $\mathfrak{A}$ induced by the embedding of Ω in $\mathfrak{A}$).*

7. The description of the construction of semigroups of the class $\mathfrak{K}$, together with the theorem on their isomorphism (Theorem 5), gives a description of semigroups of the class $\mathfrak{K}$.

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Note: Figure translations are in progress. See original paper for figures.