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## Abstract

## Full Text

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## MATHEMATICS

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# SEPARABILITY OF COMPLETE DIRECT SUMS OF TORSION-FREE ABELIAN GROUPS OF RANK 1

*(Presented by Academician P. S. Aleksandrov on 3 November 1961)*

Let

$$G = \sum_{\alpha \in M}^{*} R_\alpha$$

be a complete direct sum of torsion-free abelian groups $R_\alpha$ of rank 1 ([1]). We shall call the groups $R_\alpha$ the components of the group $G$. The type of the group $R_\alpha$ will be denoted by $|R_\alpha|$.

A torsion-free abelian group is called **completely decomposable** if it is the (ordinary) direct sum of torsion-free abelian groups of rank 1. The group

$$G = \sum_{\alpha \in M}^{*} R_\alpha$$

is completely decomposable if and only if all the groups $R_\alpha$, with the exception, possibly, of only finitely many of them, are isomorphic to the additive group of all rational numbers ([5]), i.e., a complete direct sum of torsion-free abelian groups of rank 1 is completely decomposable only in the trivial case.

A generalization of the notion of a completely decomposable group is the notion of a separable group. A torsion-free abelian group is called **separable** if each of its finite systems of elements is contained in some completely decomposable direct summand of this group. All properties of separable groups concerning finite systems of elements or subgroups of finite rank coincide with the corresponding properties of completely decomposable groups.

In Fuchs' s book ([2]) the following question was posed (problem 25): what are all those complete direct sums of torsion-free abelian groups of rank 1 that are separable groups?

I have obtained an answer to this question. It may be formulated in the form of the following theorem.

**Theorem.** The group

$$G = \sum_{\alpha \in M}^{*} R_\alpha,$$

where all $R_\alpha$ are torsion-free abelian groups of rank 1, is separable if and only if the following conditions are satisfied:

1) every decreasing sequence of types of components $R_\alpha$ of the group $G$ terminates after finitely many steps;
2) there does not exist an infinite number of pairwise incomparable types of groups $R_\alpha$ $(\alpha \in M)$;
3) if, in the characteristic determining the type

$$\mathfrak{a}_0 = |R_{\alpha_0}|$$

of the component $R_{\alpha_0}$ of the group $G$, on a set of places $N$ corresponding to an infinite number of distinct prime numbers there stands neither 0 nor $\infty$, then there exists only a finite number of components $R_\alpha$ of the group $G$ whose types are greater than or equal to $\mathfrak{a}_0$ and are determined by characteristics having finite numbers on infinite subsets of the set $N$.

Thus, the separability of the group

$$G = \sum_{\alpha \in M}^{*} R_\alpha$$

is completely characterized by the types of its components $R_\alpha$.

Conditions necessary and sufficient for the separability of a complete direct sum of torsion-free abelian groups of rank 1 were also published in the papers ([3], [4]). However, the conditions given (without proof) in pa-

in the paper ([3]), in fact are not sufficient for the separability of the group

$$G = \sum_{\alpha \in M}^{*} R_\alpha,$$

as was also noted in the paper ([4]).

The necessary and sufficient conditions for the separability of the group

$$G = \sum_{\alpha \in M}^{*} R_\alpha,$$

formulated in Theorem 5.2 of the paper ([4]), are as follows.

Denote by $\Omega(G)$ the set of all distinct types of the components $R_\alpha$ of the group

$$G = \sum_{\alpha \in M}^{*} R_\alpha,$$

and by $\Omega_{(0,\infty)}(G)$ the subset of the set $\Omega(G)$ consisting of all those types $\alpha \in \Omega(G)$ that are determined by characteristics having at each place 0 or $\infty$; put

$$\Omega_*(G) = \Omega(G) \setminus \Omega_{(0,\infty)}(G).$$

For the separability of the group $G = \sum_{\alpha \in M}^{*} R_\alpha$ it is necessary and sufficient that the following conditions be satisfied:

a) for every set $\Omega' \subseteq \Omega(G)$ and every choice of characteristics $\chi_\alpha$ in the types $\alpha$ constituting the set $\Omega'$, there exists a finite subset $\Omega'_0 \subseteq \Omega'$ such that

$$\bigcap_{\alpha \in \Omega'} \chi_\alpha = \bigcap_{\alpha \in \Omega'_0} \chi_\alpha;$$

b) in $\Omega(G)$ there is no infinite number of pairwise incomparable types;

c) for each $\alpha \in \Omega_*(G)$ there exists only a finite number of distinct components $R_\alpha$ of the group $G$ having type $|R_\alpha| = \alpha$.

Unfortunately, the sufficiency of conditions a)–c) for the separability of the group

$$G = \sum_{\alpha \in M}^{*} R_\alpha$$

was not proved in the paper ([4]), since the assertion contained erroneously in the proof of Theorem 5.2 of that paper–that under conditions a), b) the set $\Omega_*(G)$ is necessarily finite–is false. Also erroneous is the assertion contained in Theorem 5.1 of the paper ([4]) that the separability of the group

$$G = \sum_{\alpha \in M}^{*} R_\alpha$$

always entails the finiteness of the set $\Omega_*(G)$.

Indeed, let us partition the set $P$ of all prime numbers into an infinite number of (disjoint) infinite subsets $P_1, \dots, P_n, \dots$, and form characteristics $\chi_1, \dots, \chi_n, \dots$, where in the characteristic $\chi_i$ $(i = 1, 2, \dots)$ at all places corresponding to the prime numbers belonging to the sets $P_1, \dots, P_i$ there stands $\infty$; at the places corresponding to the prime numbers of the set $P_{i+1}$ there stand numbers distinct from 0 and $\infty$, and at all remaining places there stand zeros. Then take torsion-free abelian groups $R_{\alpha_1}, \dots, \dots, R_{\alpha_n}, \dots$, where $R_{\alpha_i}$ $(i = 1, 2, \dots)$ has rank 1, and the type $|R_{\alpha_i}|$ is determined by the characteristic $\chi_i$, and consider the group

$$G = \sum_{i=1,2,\dots}^{*} R_{\alpha_i}.$$

The group $G$ is separable, conditions a)–c) are satisfied for it, but the set $\Omega_*(G) = \Omega(G)$ is infinite.

It can be proved, however, that conditions a)–c) of Theorem 5.2 of the paper ([4]) are in fact equivalent to conditions 1)–3) of the theorem stated above.

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## REFERENCES

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*Note: Figure translations are in progress. See original paper for figures.*