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Abstract

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THE FIRST BOUNDARY-VALUE PROBLEM FOR QUASILINEAR PARABOLIC EQUATIONS OF SECOND ORDER OF GENERAL FORM

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In papers ([1,2]) (see the survey ([3])) we investigated parabolic equations of second order "with divergent principal part" ; in particular, for them the unique solvability in the large of the first boundary-value problem was proved. As shown in ([1,3]), all the results of these papers, namely those which concern generalized solutions, will not be valid for general quasilinear equations

$$u_t - \sum_{i,j=1}^{n} a_{ij}(x,t,u,u_{x_k})u_{x_i x_j} + a(x,t,u,u_{x_k}) = 0. \qquad (1)$$

However, statements on unique solvability in the large of the first boundary-value problem in classes of smooth functions will apparently hold for them in the same generality as for equations with divergent principal part.

The natural "restrictions" on the functions $a_{ij}(x,t,u,p_k)$ and $a(x,t,u,p_k)$, apart from a certain smoothness of these functions, are as follows*:

a) for $(x,t) \in \overline{Q}_T = \overline{\Omega} \times [0 \le t \le T]$ and arbitrary u

$$a(x,t,u,0) \ge -b_1u^2 - b_2, \qquad b_i = \text{const} \ge 0,$$

$$\sum_{i,j=1}^{n} a_{ij}(x,t,u,0)\xi_i\xi_j \ge 0;$$

b) for $(x,t) \in \overline{Q}_T$ and arbitrary u, p_k

$$\nu(|u|)(1+p)^{m-2}\sum_{i=1}^{n}\xi_i^2 \le a_{ij}(x,t,u,p_k)\xi_i\xi_j \le \mu(|u|)(1+p)^{m-2}\sum_{i=1}^{n}\xi_i^2,$$

$$|a| + \left|\frac{\partial a}{\partial u}\right| + \left|\frac{\partial a}{\partial p_k}\right|(1+p) + \left|\frac{\partial a}{\partial x_k}\right| + \left|\frac{\partial a_{ij}}{\partial u}\right|(1+p)^2 +$$

$$+\left|\frac{\partial a_{ij}}{\partial p_k}\right|(1+p)^3 + \left|\frac{\partial a_{ij}}{\partial x_k}\right|(1+p)^2 \leq \mu(|u|)(1+p)^m,$$

where m is any number.

Moreover, if there exists a classical solution of equation (1) (we shall speak of solutions from $C^{\alpha,\alpha/2}_{2,1}(\overline{Q}_T)$) satisfying the boundary and initial conditions

$$u|_S = 0, \qquad u|_{t=0} = \varphi(x), \tag{2}$$

then it is necessary that

$$\varphi|_S = 0, \qquad \left[-\sum a_{ij}(x,0,\varphi,\varphi_{x_k})\varphi_{x_i x_j} + a(x,0,\varphi,\varphi_{x_k})\right]_S = 0. \tag{3}$$

* We shall use the notation from paper ([1]); in particular, ν, μ are everywhere positive,

$$p = \left(\sum_{k=1}^{n} p_k^2\right)^{1/2}.$$

Here, as in all our papers ([1–6]), we assume that the Hölder exponent α belongs to $(0,1)$. True, some of the theorems are also valid for $\alpha = 1$.

We have proved the following theorem:

Theorem 1. *Suppose that:*

1) Conditions a), b) are satisfied, where the inequalities in b) hold only for

$$|u| \leqslant M = \min_{b>b_1} e^{bT}\left[\max_{\Omega}|\varphi| + \sqrt{\frac{b_2}{b-b_1}}\right].$$

2) For the same x, t, u, p_k as in b), the inequalities

$$\left|\frac{\partial a_{ij}(x,t,u,p_k)}{\partial u}\right| \leqslant \varepsilon(1+p)^{m-2} + P(p)(1+p)^{m-2}, \tag{4}$$

$$-\frac{\partial a(x,t,u,p_k)}{\partial u} \leqslant \varepsilon(1+p)^m + P(p)(1+p)^m \tag{5}$$

hold with some, generally speaking, small $\varepsilon > 0$, determined only by the data of the problem, and with a function $P(p) \to 0$ as $p \to \infty$.

3) For $(x,t) \in \overline{Q}_T$, $|u| \leqslant M$, $p \leqslant M_1$ (where M_1 is determined only by the data of the problem), the functions $a_{ij}(x,t,u,p_k)$, $a(x,t,u,p_k)$ are continuous and satisfy the Hölder condition in x,t,u,p_k with exponents $\beta, \beta/2, \beta, \beta$, respectively.

4) $S \in C_{2,\beta}$, $\varphi \in C_{2,\beta}(\overline{\Omega})$, and (3) holds.

If these conditions are satisfied, there exists a unique classical solution $u(x,t)$ of problem (1), (2), more precisely, a solution of the class $C_{2,1}^{\beta,\beta/2}(\overline{Q}_T)$.

Of all the conditions of this theorem, only the requirement of smallness in (4), (5) is caused, probably, not by the substance of the matter. For an equation with divergent principal part for $m = 2$, as shown in ([1,2]), it is superfluous.

Without any other assumptions, apart from the condition of parabolicity and smoothness, the following theorem is proved:

Theorem 2. *Let $u(x,t)$ be a solution of equation (1) from $C_{2,1}(\overline{Q}_T)$, and let equation (1) be parabolic on it, i.e., for $(x,t) \in \overline{Q}_T$,*

$$a_{ij}(x,t,u(x,t),u_{x_k}(x,t))\,\xi_i\xi_j \geqslant \nu \sum_{i=1}^{n} \xi_i^2.$$

Let $a_{ij}(x,t,u,p_k)$, $a(x,t,u,p_k)$ be differentiable with respect to x,t,u,p_k, and let all their first-order partial derivatives, for $u = u(x,t)$, $p_k = u_{x_k}(x,t)$, be bounded for $(x,t) \in \overline{Q}_T$ by some constant μ. Then the Hölder norms $|u_{x_i}|_{\alpha,Q_T}$, with some $\alpha > 0$, of the derivatives u_{x_i} are estimated in terms of

$$\max_{Q_T,k} |u, u_{x_k}|, \quad \sum_{k=1}^{n} |u_{x_k}(x,0)|_{\beta,\Omega}, \quad \nu, \quad \mu$$

of only the conditions just formulated, the Hölder norms of the tangential derivatives of u on Γ, and the norms in $C_{2,0}$ of the functions defining the boundary S.

Interior estimates $|u_{x_i}|_{\alpha,Q'_T}$ depend only on $\max_{Q_T,k} |u, u_{x_k}|$, ν, μ, and the distance of Q'_T from the boundary of Q_T.

This theorem, together with Friedman' s results ([7]) on linear parabolic equations, makes it possible to estimate the Hölder norms for $u_t, u_{x_i x_j}$ and higher derivatives of u in terms of $\max_{Q_T,i} |u, u_{x_i}|$ and the corresponding norms of the boundary and initial values, under the sole assumption of parabolicity of (1) and the corresponding smoothness of the functions u, a_{ij}, a.

In turn, for a solution $u(x,t)$ of problem (1), (2) belonging to $C_{2,1}(\overline{Q}_T)$, one can estimate $\max_{Q_T} |u_{x_i}|$ in terms of the modulus of continuity of the function $u(x,t)$ and the data of the problem, also only under the "natural assumptions"b).

For this purpose we modify the estimate of $\max_{Q_T}|u_{x_i}|$ in ([4,5]), whose method required the existence, for the solution, of the derivatives u_{tx_i} and $u_{x_ix_jx_k}$.

We shall give a scheme for carrying out the estimates formulated here for solutions $u(x,t)$ of equation (1) (we note that all interior estimates, with the possible exception of $\max_{Q_T}|u|$, do not depend on the boundary and initial conditions for u and, consequently, are applicable to all boundary-value problems).

First we estimate $\max_S|u_{x_i}|$, as was done in ([4], [5]). We note that, for a domain Ω of arbitrary connectivity, in order to estimate $\max|u_{x_i}|$ on some piece S_1 of the boundary S, the domain must first be transformed so that, with respect to the image of this piece S_1, the transformed domain is situated in the manner required in § 3 of ([5]).

To estimate $\max|u_{x_i}|$ in all of Q_T, we introduce a certain new function v by means of the equality $u = \psi(v)$. For $w = \sum_{i=1}^n v_{x_i}^2$, by virtue of (1) we obtain a relation of the type

$$\frac{\partial w}{\partial t} - \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial w}{\partial x_j}\right) + \cdots = 0. \tag{6}$$

We multiply it by $w-\lambda$ and integrate first over the set $A_\lambda(t)$ of points $x \in \Omega$ where $w(x,t) > \lambda$, and then in t from 0 to T. All the subsequent arguments are carried out with this equality, taking λ sufficiently large and close to the assumed maximum of w. This leads to an estimate of $\max_{Q_T}|u_{x_i}|$.

After this we estimate $\max_{Q_T}|u_{x_i}|$. For this purpose we introduce the auxiliary function

$$v = u_t + \mu\sum_{k=1}^n u_{x_k}^2.$$

For it, from (1) we derive a relation of the form

$$\frac{\partial v}{\partial t} - \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial v}{\partial x_j}\right) + \cdots = 0.$$

We multiply it by $v-\lambda$ and integrate first over the set $\{v(x,t) > \lambda\}$, and then in t from 0 to T. The relation thus obtained is the basis for estimating $\max_{Q_T}|u_t|$. The numbers λ and μ are chosen sufficiently large.

After the estimate of $\max_{Q_T}|u_t|$ has been obtained, we consider equation (1) as elliptic,

$$a_{ij}u_{x_ix_j} = u_t + a = F \tag{7}$$

with bounded right-hand side F. In ([6]), for solutions $u \in W_2^2(\Omega) \cap C_1^1(\Omega)$ of such equations, an estimate of $|u_{x_i}|_{\alpha,\Omega}$ was obtained in terms of $\max_{\Omega,k} |u, u_{x_k}|$ and quantities known in the problem. Knowing $\max_{Q_T} |u_t|$ and $\max_{0 \le t \le T} |u_{x_i}|_{\alpha,\Omega}$, it is not difficult to obtain an estimate of $|u_{x_i}|_{\beta,Q_T}$ (see, for example, § 2 of ([2])) and thereby to complete the full cycle of estimates.

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Note: Figure translations are in progress. See original paper for figures.