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Abstract

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AN ESTIMATE OF A SUM WITH PRIME NUMBERS

(Presented by Academician I. M. Vinogradov, 19 XII 1961)

In paper (1), p. 305, I. M. Vinogradov pointed out the possibility of estimating the sum

$$S = \sum_{p \le N} \chi(f(p)),$$

where χ is a nonprincipal character modulo the prime q, and $f(x)$ is an integral polynomial. In the present note, by I. M. Vinogradov' s method, an estimate is derived for a sum of the more general form

$$\sum_{p \le N} \chi(R_1(p)) \exp\left(\frac{2\pi i}{q} R_2(p)\right),$$

where R_1, R_2 are rational functions modulo q. The paper uses A. Weil' s estimates (2) in a strengthened form.

Let f_1, g_1, f_2, g_2 be polynomials with integral rational coefficients, with $(f_i, g_i) = 1$, and let the leading coefficients of the polynomials f_1, g_1 be equal to 1; let q be a prime $> q_0$, where q_0 is a sufficiently large positive integer depending on f_i, g_i; let χ be a nonprincipal character modulo q; $\chi(x/y) = \chi(xy')$, where $yy' \equiv 1 \pmod{q}$, if $q \nmid y$; $\chi(x/y) = 0$, if $q \mid y$; $R_i(x)$ are "rational functions modulo q," defined for every integer a, $q \nmid g_i(a)$, from the condition $R_i(a)g_i(a) \equiv f_i(a) \pmod{q}$; over the prime field Π_q they define the ratios

$$\frac{f_i(x)}{g_i(x)} \in \Pi_q$$

for $g_i(x) \ne 0,\ x \in \Pi_q$. We shall also put $R_i(a) = 0$, if $q \mid g_i(a)$. Let

$$\varphi(x) = \chi(R_1(x)) \exp\left(\frac{2\pi i}{q} R_2(x)\right);$$

$\bar{z}$ is the value complex conjugate to z; ε is an arbitrary small positive number; the symbols $A = O(B)$ or $A \ll B$ mean that $|A| \le c|B|$, where c is a constant depending on f_i, g_i, ε.

Theorem. Let f_1, g_1, f_2, g_2 be polynomials with integral rational coefficients such that, if

$$\frac{f_2}{g_2} = ax + b,$$

then

$$\frac{f_1}{g_1} \neq x,\ \frac{1}{x},\ \text{const},$$

and the polynomial $f_1 g_1$ has no multiple roots; let q be a prime number, χ a nonprincipal character modulo q; let p run through the consecutive prime numbers; and let N be a positive integer.

Then, for any independent $\varepsilon > 0,\ \frac{1}{6} \geq \varepsilon_0 > 0$, we have:

$$S = \sum_{p \leq N} \varphi(p) = \sum_{p \leq N} \chi(R_1(p)) \exp\left(\frac{2\pi i}{q} R_2(p)\right)$$
$$= O\left(N^{1+\varepsilon} q^{\varepsilon} \sqrt{\frac{1}{\sqrt{q}} + \frac{\sqrt{q}}{N^{1/2+\varepsilon_0}} + N^{-\varepsilon_0}}\right). \tag{1}$$

The proof of the theorem is based on six lemmas.

Lemma A. Let $R_1(x), R_2(x)$ be rational functions modulo q, and let χ be a nonprincipal character modulo q, with at least one of the conditions being satisfied: 1) $\chi(R_1(x)) \neq \text{const}$; 2) $R_2(x) \neq \text{const}$. Then

$$\sum_{a=0}^{q-1} \varphi(a) \ll \sqrt{q}.$$

The proof is obtained by combining the ideas of Hasse’ s work (3) with A. Weil’ s theorem on the zeros of the zeta-function of algebraic curves.

In all that follows it is assumed that the polynomials f_i, g_i satisfy the conditions of the theorem.

Lemma 1. Let N, Y, X be integers, $Y < 0,\ 0 < X < q$; a, b integers, $(a, q) = 1$; d_1, d_2 integers independently running through the interval $(N, N+Y)$. Then:

$$1)\quad \sum_{x=0}^{q-1} \varphi(ax) \exp\left(\frac{2\pi i}{q} bx\right) \ll \sqrt{q},$$

$$1')\quad \sum_{x=N}^{N+X} \varphi(ax) \ll \sqrt{q} \ln q$$

uniformly with respect to a, b, N, X;

$$2) \quad \sum_{x=0}^{q-1} \varphi(ad_1x)\overline{\varphi(ad_2x)} \exp\left(\frac{2\pi i}{q}bx\right) \ll \sqrt{q},$$

$$2') \quad \sum_{x=N}^{N+X} \varphi(ad_1x)\overline{\varphi(ad_2x)} \ll \sqrt{q}\ln q$$

uniformly with respect to a, b, d_1, d_2, N, X for all pairs d_1, d_2, with the possible exception of

$$\ll Y + \frac{Y^2}{q}$$

pairs.

Proof. Estimate 1) follows from Lemma A. 1′) follows from 1) and a well-known lemma of I. M. Vinogradov.

If $(d_1d_2, q) = 1$, then the conditions of Lemma A for the sum 2) can fail to hold only for those pairs d_1, d_2 for which the following congruences of polynomials modulo q hold simultaneously:

$$f_1(ad_1x)g_1(ad_2x) = cf_1(ad_2x)g_1(ad_1x) \pmod{q},$$

$$f_2(ad_1x)g_2(ad_2x) - f_2(ad_2x)g_2(ad_1x) = (c_1 + b_1x)g_2(ad_1x)g_2(ad_2x) \pmod{q},$$

where c, c_1, b_1 are integers. It is easy to show that on the interval $(N, N+Y)$ the number of such pairs is

$$\ll Y + \frac{Y^2}{q}.$$

Therefore 2), 2′) are obtained in the same way as 1), 1′).

Lemma 2. Let M, N, X, Y be integers, $X > 0,\ Y > 0,\ (a, q) = 1$;

$$S_a = \sum_{x=M}^{M+X} \sum_{y=N}^{N+Y} \xi(x)\eta(y)\varphi(axy), \qquad 0 \le \xi(x) \le \alpha, \quad 0 \le \eta(y) \le \beta.$$

Then

$$S_a \ll \alpha\beta q^{\varepsilon_1} XY\sqrt{\frac{1}{\sqrt{q}}+\frac{\sqrt{q}}{X}+\frac{1}{Y}}$$

uniformly with respect to a.

Proof.

$$|S_a|^2 \ll \alpha^2\beta^2 X \sum_{d_1=N}^{N+Y}\sum_{d_2=N}^{N+Y}\sum_{x=M}^{M+Y}\varphi(ad_1x)\overline{\varphi(ad_2x)};$$

with the exception of

$$\ll Y+\frac{Y^2}{q}$$

pairs d_1, d_2, by 2), 2′) of Lemma 1 the inner sum will be

$$\ll \frac{X}{q}\sqrt{q}+\sqrt{q}\ln q.$$

Consequently,

$$|S_a|^2 \ll \alpha^2\beta^2 X^2Y^2 q^{\varepsilon_1}\left(\frac{1}{\sqrt{q}}+\frac{\sqrt{q}}{X}+\frac{1}{Y}\right).$$

Lemma 3. Let $cu < u' \leq 2u,\ 1 < c < 2,\ (a, q) = 1;$

$$S_a = \sum_x\sum_y \xi(x)\eta(y)\varphi(axy),$$

where the summation extends over the domain

$$xy \leq N, \qquad u < x \leq u'; \qquad 0 \leq \xi(x) \ll N^{\varepsilon_2}, \quad 0 \leq \eta(y) \ll N^{\varepsilon_2}.$$

Then

$$S_a \ll N^{1+\varepsilon_2} q^{\varepsilon_3}\sqrt{\frac{1}{\sqrt{q}}+\frac{\sqrt{q}}{u}+\frac{u'}{N}}$$

uniformly with respect to a.

Lemma 3 is proved by the "exhaustion" method on the basis of Lemma 2, in the same way as Lemma 4 in the work of I. M. Vinogradov (^{1}), p. 219.

Lemma 4. Let x, y, m run through values belonging to three increasing sequences; $1 < u_1 < u_2 \leqslant N$,

$$S = \sum_x \sum_y \sum_m \varphi(xym),$$

where the summation extends over the domain

$$u_1 < x \leqslant u_2, \qquad xym \leqslant N, \qquad (x, y) = 1.$$

Then

$$S \ll N^{1+\varepsilon_4} q^{\varepsilon_4} \sqrt{\frac{1}{\sqrt{q}} + \frac{\sqrt{q}}{u_1} + \frac{u_2}{N}}.$$

Proof. We find $S = \sum_d \mu(d) S_d$, where d runs through the positive integers simultaneously dividing the numbers in at least one of the possible pairs x, y;

$$S_d = \sum_{x'} \sum_{y'} \sum_m \varphi(d^2 x' y' m),$$

where x', y' run through the quotients obtained by dividing by d the values x, y divisible by d, and the summation extends over the domain:

$$\frac{u_1}{d} < x' \leqslant \min\left(\frac{u_2}{d}, \frac{N}{d^2}\right), \qquad x' y' m \leqslant \frac{N}{d^2}.$$

This domain can be subdivided into $\ll \ln N$ domains of the form

$$\frac{u}{d} < x' \leqslant \frac{u'}{d}, \qquad x' y'' \leqslant \frac{N}{d^2}, \tag{2}$$

where $u_1 \leqslant u < u' \leqslant \min\left(2u, u_2, \frac{N}{d}\right)$, $y'' = y'm$. Since the number $\eta(y'')$ of pairs y', m with the condition $y'm = y''$, for $y'' \leqslant N$, will be $\ll N^{\varepsilon_2}$, putting $\xi(x') = 1$ if x' is an admissible value, and $\xi(x') = 0$ otherwise, we may write the part S'_d of the sum S_d corresponding to one of the distinguished domains in the form

$$S'_d = \sum_{x'} \sum_{y''} \xi(x') \eta(y'') \varphi(d^2 x' y''),$$

where the summation extends over the domain (2).

If $(d, q) = 1$, then, applying Lemma 3, we obtain

$$S_d \ll \left(\frac{N}{d^2}\right)^{1+\varepsilon_3} q^{\varepsilon_3} \sqrt{\frac{1}{\sqrt{q}} + \frac{\sqrt{q}\,d}{u} + \frac{u' d^2}{N d}} \ll \frac{N^{1+\varepsilon_3} q^{\varepsilon_3}}{d} \sqrt{\frac{1}{\sqrt{q}} + \frac{\sqrt{q}}{u_1} + \frac{u_2}{N}}.$$

If $(d, q) = q$, then, estimating S_d trivially, we have

$$S_d \ll N^{\varepsilon_2} \sum_{x'} \sum_{y'' \leqslant \frac{N}{d^2}} 1 \ll \frac{N^{1+\varepsilon_3}}{d^2}.$$

Since all values of d simultaneously dividing the numbers in one of the possible pairs x, y do not exceed $\sqrt{N}$, we have

$$S \leqslant \sum_{d \leqslant \sqrt{N}} |S_d| = \sum_{\substack{(d,q)=1 \\ d \leqslant \sqrt{N}}} |S_d| + \sum_{\lambda \leqslant \frac{\sqrt{N}}{q}} |S_{\lambda q}| \ll$$

$$\ll N^{1+\varepsilon_4} q^{\varepsilon_4} \sqrt{\frac{1}{\sqrt{q}} + \frac{\sqrt{q}}{u_1} + \frac{u_2}{N}} + \sum_{\lambda \leqslant \frac{\sqrt{N}}{q}} \frac{N^{1+\varepsilon_3}}{q^2 \lambda^2} \ll N^{1+\varepsilon_4} q^{\varepsilon_4} \sqrt{\frac{1}{\sqrt{q}} + \frac{\sqrt{q}}{u_1} + \frac{u_2}{N}}.$$

Lemma 5. Let $0 < \varepsilon_0 \leqslant \frac{1}{6}$, $0 < h < \varepsilon_0$; let N be a sufficiently large positive integer; and let P be the product of certain primes not exceeding $N^{1/2-\varepsilon_0}$.

Then the divisors d of the number P not exceeding N can be distributed

among $< D$ classes, where $D = (\ln N)^{\ln\ln N/\ln(1+h)}$, and for all d of one and the same class $\mu(d)$ retains a constant value.

Some of these classes include only values d satisfying $d \leqslant N^{1/2+h}$. For each of the remaining classes there exists a positive integer H and two increasing sequences (x) and (y) of positive integers satisfying $N^{1/2} < x \leqslant N^{1-\varepsilon_0+h}$ such that all the numbers of the class, each H times, are obtained if from all products we choose only those satisfying the conditions

$$xy \leqslant N, \qquad (x, y) = 1.$$

Lemma 5 is an insignificant modification of a special case of Lemma 6 in the paper of I. M. Vinogradov (1), p. 221.

Proof of the theorem. Let P be the product of all primes p distinct from q and satisfying $p \leqslant N^{1/2-\varepsilon_0}$; let Q be the product of all primes p distinct from q and satisfying $N^{1/2-\varepsilon_0} < p \leqslant N$.

We find (p_1, p_2 run through prime numbers):

$$S = \sum_{d|P} \sum_{\substack{md\leqslant N\\(m,q)=1}} \mu(d)\varphi(md) - \sum_{\substack{p_1|Q\\p_1p_2\leqslant N}} \sum_{\substack{p_2|Q\\(p_1,p_2)=1}} \varphi(p_1p_2) + O(N^{1/2}). \qquad (3)$$

Taking some h satisfying $0 < h < \varepsilon_0$, we divide all values d into $\leqslant N^{\varepsilon_5}$ classes, as indicated in Lemma 5. First estimate the first double sum on the right-hand side of equality (3). The part S' of this sum corresponding to one of the classes for which $d \leqslant N^{1/2+h}$ is equal to

$$S' = \sum_{d} \sum_{md\leqslant N} \mu(d)\varphi(md) - \sum_{d} \sum_{\substack{md\leqslant N\\(m,q)=q}} \mu(d)\varphi(md) = S'_1 - S'_2.$$

By 1), 1′) of Lemma 1 we have:

$$S'_1 \ll \sum_{\substack{d\leqslant N^{1/2+h}\\q\nmid d}} \left| \sum_{\substack{m\leqslant N/d\\(m,q)=1}} \varphi(md) \right| \ll \sum_{d<N^{1/2+h}} \left(\frac{N}{dq}\sqrt{q} + \sqrt{q}\ln q \right) \ll$$

$$\ll \frac{N^{1+\varepsilon_5}}{\sqrt{q}} + q^{\varepsilon_5} N^{1/2+h}\sqrt{q} \ll N^{1+\varepsilon_5+h} q^{\varepsilon_5} \left(\frac{1}{\sqrt{q}} + \sqrt{\frac{q}{N}} \right);$$

$$S'_2 \ll \sum_{d\leqslant N} \sum_{m\leqslant N/(dq)} 1 \ll \frac{N^{1+\varepsilon_5}}{q}.$$

Consequently,

$$S' \ll N^{1+\varepsilon_5+h} q^{\varepsilon_5} \left(\frac{1}{\sqrt{q}} + \sqrt{\frac{q}{N}} \right).$$

The part S'' of the sum under consideration corresponding to one of the remaining classes, according to Lemmas 5 and 4 ($u_1 = N^{1/2}$, $u_2 = N^{1-\varepsilon_0+h}$), will be

$$\ll N^{1+\varepsilon_4} q^{\varepsilon_4} \sqrt{\frac{1}{\sqrt{q}} + \sqrt{\frac{q}{N}} + \frac{N^{1-\varepsilon_0+h}}{N}} \ll$$

$$\ll N^{1+\varepsilon_4+h} q^{\varepsilon_4} \sqrt{\frac{1}{\sqrt{q}} + \sqrt{\frac{q}{N}} + N^{-\varepsilon_0}}.$$

Finally, the second double sum on the right-hand side of equality (3), according to Lemma 4 ($u_1 = N^{1/2-\varepsilon_0},\ u_2 = N^{1/2+\varepsilon_0},\ (x) = (p_1),\ (y) = (p_2),\ (m) = (1)$), will be

$$\ll N^{1-\varepsilon_4} q^{\varepsilon_4} \sqrt{\frac{1}{\sqrt{q}} + \frac{\sqrt{q}}{N^{1/2-\varepsilon_0}} + N^{-1/2+\varepsilon_0}}.$$

Considering that

$$\frac{1}{\sqrt{q}} + \sqrt{\frac{q}{N}} < 1,$$

and that $\varepsilon_4, \varepsilon_5, \varepsilon_6, h$ are sufficiently small, we obtain (1).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.