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Abstract

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ON COMPACTIFICATIONS OF COMPLETELY REGULAR SPACES

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In this note an example is constructed of a completely regular space that is not compactifiable in any normal space, and a theorem on compactifiability in bicompacta is proved, generalizing one result of I. L. Raukhvarger ([1]).

Take the product $P' = I \times W_\alpha$; I is an interval of the line, and W is the set of all ordinal numbers $\leq \alpha$, where α is any, but fixed once and for all, transfinite number such that no sequence of cardinality $\leq 2^c$ is cofinal in it. From P' we remove the points whose first coordinate is irrational and whose second is equal to α; we obtain the space P. The space P is completely regular and peripherally bicompact. We shall prove that P is not compactifiable in any normal space.

Consider the auxiliary space $Q = (\Xi \times W_\alpha) \setminus (\sigma, \alpha)$, where Ξ is a countable sequence with limit point σ, and W_α is the set of all transfinite numbers up to α, inclusive.

Lemma. *Let a compactification $f : Q \to X$ be given, where X is completely regular. It is assumed here that the image of the set of all points of Q whose second coordinate is equal to α has no limit point in X. Then f is a homeomorphism.*

Suppose that f is not a homeomorphism. Then $[fA] \neq fA$, where A is some closed subspace of Q. Take a point $\xi \in [fA] \setminus fA$ and an arbitrary neighborhood $O\xi$ of it. We shall show that $[f^{-1}O\xi] \ni (\sigma, \alpha)$; here the closure is taken in the space Q augmented by the point (σ, α); the so augmented Q we denote by $\overline{Q}$. We argue by contradiction. Let $(\sigma, \alpha) \notin [f^{-1}O\xi]$. Then $A' = A \cap [f^{-1}O\xi]_{\overline{Q}}$ is bicompact. The bicompact fA' coincides with $fA \cap [O\xi]$, since $f[f^{-1}O\xi] = [O\xi]$. But $\xi \in [fA] \setminus fA$, and therefore $fA \cap [O\xi]$ cannot be bicompact. Thus $(\sigma, \alpha) \in [f^{-1}O\xi]_{\overline{Q}}$. Moreover, we shall show that $[f^{-1}O\xi]$ contains a countable subset of the set of points with second coordinate α. Indeed, since $(\sigma, \alpha) \in [f^{-1}O\xi]_{\overline{Q}}$, the open set $f^{-1}O\xi$ contains a set V cofinal in the set of all points with first coordinate σ. The set $[OV]$, where OV is an arbitrary neighborhood of V, contains all points with second coordinate α, except, perhaps, finitely many of them. In particular, this property is possessed by $[f^{-1}O\xi]$. Since $O\xi$ is an arbitrary neighborhood of ξ, ξ is a limit point for the image of the set of points with second coordinate α. This contradicts the hypothesis of the lemma. Thus f is a homeomorphism.

We return to the space P. Suppose that there exists a compactification $f: P \to X$, where X is normal. Denote by G the set of points of P with second coordinate α. The cardinality $|fG| \le 2^c$, since G is countable. Every countable subset $N \subseteq fG$ has at least one limit point. Indeed, suppose that some $N \subseteq fG$ has no limit points in X. Then $f^{-1}N \subset (I, \alpha)$ has no limit points in P. Take in P a closed subspace R such that R is homeomorphic to Q and

$R \cap (I, \alpha) \subseteq f^{-1}N$. By the lemma, the mapping f on R is a homeomorphism. The set fR is closed in X and is not normal; hence X is a nonnormal space. Thus every $N \subseteq fG$ has a limit point. Further, $[fG]$ is countable, since G is not embedded in an everywhere dense subset of a countable compactum, because a countable compactum has an isolated point, whereas G has none. It is not hard to show that $D = [fG] \setminus G$ is everywhere dense in $[fG]$ and, in particular, D is not closed in X. The set $f^{-1}[fG] = f^{-1}D \cup G$ is a closed set, but then $f^{-1}D$ is closed and bicompact, since the cardinality of $f^{-1}D \le 2^c$, and α was chosen in a special way. But then D is also bicompact, which contradicts the fact that D is not closed in the space X. Thus P is not embedded in a normal space.

I. Raukhvarger proved [1] the theorem: if a countable number of points is removed from a compactum, then the remaining space is embedded in a compactum.

Here the following will be proved.

Theorem. *Let a bicompactum B be given which is a product of compacta,*

$$B = \prod_{\alpha}^{\tau} K_\alpha.$$

If a countable number of points is removed from B, then the remaining space is embedded in a bicompactum.

It is necessary to prove that $B' = B \setminus A$ is embedded in a bicompactum, where A is a countable subset, $A = \{a_i\},\ i = 1, 2, \ldots$. We may suppose that no point a_i is isolated in B: the union of all points of A isolated in B is an open set V_1, so that $B \setminus V_1$ is a bicompactum; if among $A \cap V_1$ in the bicompactum $B \setminus V_1$ there are isolated points, then in any case after a countable number of such steps we shall get rid of them, and everything will reduce to the case of a space without isolated points.

The point a_i has coordinate $a_{i\alpha}$ with respect to the compactum K_α. Take the closed $\frac{1}{i}$-neighborhood $Oa_{i\alpha}$ of the point $a_{i\alpha}$ in K_α (i.e. the set of all points at distance from $a_{i\alpha}$ not greater than $\frac{1}{i}$). In the set $\prod_\alpha^\tau Oa_{i\alpha}$ choose any point ξ_i distinct from a_i. In choosing ξ_i for different i, we shall ensure that they are pairwise distinct.

We now construct a decomposition of the bicompactum B: the elements of the decomposition will be the pairs of points (a_i, ξ_i) and the individual points of the bicompactum B. We shall prove that this decomposition is continuous. Let U_α be a one-index neighborhood of some element r of the decomposition. From Raukhvarger' s theorem for compacta it follows that in U_α one can inscribe such a one-index neighborhood V_α of the element r of the decomposition that, as soon as some element of the decomposition intersects V_α, it necessarily lies in U_α, which proves everything.

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REFERENCES

1. I. Raukhvarger, *DAN*, **66**, No. 1, 13 (1949).

Note: Figure translations are in progress. See original paper for figures.