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Abstract

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ON A BOUNDARY-VALUE PROBLEM

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Consider the differential equation

$$L(u) \equiv (-1)^{[s/2]+1}\frac{\partial^s u}{\partial t^s} + (-1)^{m+1}\sum_{|i|=|j|=m} D^i A^{ij}(x,t) D^j u + B(u) = f(x,t), \; (1)$$

$$B(u) = \sum_{|i|/2m+s_1/s<1} B^{(i,s_1)}(x,t)\,\frac{\partial^{s_1}}{\partial t^{s_1}} D^i u, \tag{2}$$

where $s \geqslant 1$, $m \geqslant 1$ are certain integers; $x = (x_1, \dots, x_n) \in \Omega_0$; Ω_0 is a domain bounded by a sufficiently smooth closed surface γ in x-space; $t \in [0,T]$; $(x,t) \in Q \equiv \Omega_0 \times (0,T)$; $\Gamma = \gamma \times [0,T]$. The functions $A^{ij}(x,t) = A^{ji}(x,t)$, $B^{(i,s_1)}(x,t)$ are sufficiently smooth in $\overline{Q}$; for simplicity we shall regard them as real-valued; $f(x,t) \in \mathcal{L}_2(Q)$;

$$i = (i_1, \dots, i_n), \quad j = (j_1, \dots, j_n); \quad |i| = i_1 + \dots + i_n; \quad D^i = \partial^{|i|}/\partial x_1^{i_1} \cdots \partial x_n^{i_n},$$

$$\sum_{|i|=|j|=m} \xi_1^{i_1} \cdots \xi_n^{i_n} A^{ij}(x,t) \xi_1^{j_1} \cdots \xi_n^{j_n} \geqslant \theta^2 > 0 \tag{3}$$

for $|\xi| = 1$, $(x,t) \in \overline{Q}$.

We shall be interested in the question of existence and uniqueness of a solution of equation (1) in the domain Q under the boundary conditions

$$\left.\frac{\partial^r u}{\partial n^r_{(x,t)}}\right|_{\Gamma} = \varphi_r(x,t), \qquad 0 \leqslant r \leqslant m-1; \tag{4}$$

$$\left.\frac{\partial^r u}{\partial t^r}\right|_{t=0} = \psi_r(x); \qquad 0 \leqslant r \leqslant k, \quad \text{if } s = 2k+1; \tag{5}$$

$$0 \leqslant r \leqslant k-1, \quad \text{if } s = 2k;$$

$$\left.\frac{\partial^r u}{\partial t^r}\right|_{t=T} = \chi_r(x), \qquad 0 \leqslant r \leqslant k-1, \tag{6}$$

where $n_{(x,t)}$ is the normal to Γ at the point (x,t);

$$\varphi_r(x,t) \in W_{t,x,2}^{(2m-r-1/2)/s,\, 2m-r-1/2}(\Gamma), \qquad \psi_r(x) \in W_{x,2}^{2m(1-r/s-1/2s)}(\Omega_0),$$

$$\chi_r(x) \in W_{x,2}^{2m(1-r/s-1/2s)}(\Omega_T), \qquad \Omega_T = \overline{Q} \cap (t=T).$$

We shall assume, in addition, that the boundary functions satisfy the natural compatibility conditions.

Let us note that in the case of odd $s = 2k+1$, instead of problem (1), (4)–(6) one may consider the problem $(\tilde{1}), (\tilde{4}), (\tilde{5}), (\tilde{6})$, where $(\tilde{1})$ differs from (1) by the sign of $\partial^s u/\partial t^s = \partial^{2k+1}u/\partial t^{2k+1}$, and the conditions $(\tilde{5})$ and $(\tilde{6})$ are obtained from (5) and (6) if the latter are interchanged.

Equation (1) for $s = 1$ is parabolic, and the problem considered for it has been well studied. For $s = 3$ the problem (1), (4)–(6) in the case,

when $B(u) \equiv 0$, was studied by A. A. Dezin; in (^{1,2}) he constructed a generalized solution of this problem*.

Theorem 1 on an a priori estimate. Let $u(x,t) \in W_{t,x,2}^{(p,q)}(Q)$, $p \geq s$, $q \geq 2m$. Then there exists a constant C, depending only on the coefficients of (1) and on the domain Q, such that

$$\|u\|^2_{W^{(p,q)}(Q)} \leq C^2 \Big(\|u\|^2_{\mathcal{L}_2(Q)} + \|Lu\|^2_{W^{(p-s,q-2m)}(Q)} +$$

$$+ \sum_{r=0}^{m-1} \|\varphi_r\|^2_{W^{(p_r,q_r)}(\Gamma)} + \sum_{r=0}^{k} \|\psi_r\|^2_{W^{q'_r}(\Omega_0)} + \sum_{r=0}^{k-1} \|\chi_r\|^2_{W^{q'_r}(\Omega_T)} \Big), \tag{7}$$

where $p_r = p(1 - r/q - 1/2q)$, $q_r = q(1 - r/q - 1/2q)$, $q'_r = q(1 - r/p - 1/2p)$.

If $p = s$, $q = 2m$, then estimate (7) can be obtained in the following way: introduce the new variable $w(x,t) = u(x,t) - z(x,t)$, where $z(x,t) \in W^{(s,2m)}(Q)$ and satisfies the boundary conditions (4)–(6) (such a function $z(x,t)$ can be constructed according to (^{3})); $w(x,t)$ satisfies the homogeneous conditions (4)–(6) (we shall call them conditions (4_0)—(6_0)) and equation (1) with right-hand side $F(x,t) = f(x,t) - L(z)$; moreover,

$$\|F\|^2_{\mathcal{L}_2(Q)} \le C_1^2 \left(\|f\|^2_{\mathcal{L}_2(Q)} + \|z\|^2_{W^{s,2m}(Q)} \right) \le$$

$$\le C_2^2 \left(\sum \|\varphi_r\|^2_{W^{(p_r,q_r)}(\Gamma)} + \sum \|\psi_r\|^2_{W^{q'_r}(\Omega_0)} + \|f\|^2_{\mathcal{L}_2(Q)} + \sum \|\chi_r\|^2_{W^{q'_r}(\Omega_T)} \right).$$

The estimate of $\|w\|^2_{W^{(s,2m)}(Q)}$ is obtained by multiplying (1) by $\partial^s w/\partial t^s$ and integrating the resulting equality over Q.

In the remaining cases $p \ge s,\ q \ge 2m$, to obtain (7) one should use a partition of unity in $\overline{Q}$, the "parametrix" of the fundamental solution for interior points of Q, or the "parametrix" of the Green function in a neighborhood of boundary points. With the aid of the "parametrix" of the Green function one can obtain an a priori estimate also under more general boundary conditions.

Theorem 2 on an a priori estimate. Let $u(x,t) \in W^{p,q}(Q)$. Then there exists a constant C, independent of $u(x,t)$, such that

$$\|u\|^2_{W^{(p,q)}(Q)} \le C^2 \left(\|u\|^2_{\mathcal{L}_2(Q)} + \|L(u)\|^2_{W^{p-s,q-2m}(Q)} + \right.$$

$$\left. + \sum \|A_r(x,t,u)\|^2_{W^{\widetilde{p}_r,\widetilde{q}_r}(\Gamma)} + \sum \|B_r(x,u)\|^2_{W^{\widetilde{q}'_r}(\Omega_0)} + \sum \|C_r(x,u)\|^2_{W^{\widetilde{q}''_r}(\Omega_T)} \right), \tag{8}$$

where

$$A_r(x,t,u) = \sum_{j=0}^{l_r} \alpha_{rj}(x,t) \frac{\partial^j u}{\partial n^j_{(x,t)}}, \qquad 0 \le r \le m-1;$$

$$B_r(x,u) = \sum_{j=0}^{m_r} \beta_{rj}(x) \frac{\partial^j u}{\partial t^j}; \qquad 0 \le r \le k \quad \text{for } s = 2k+1;$$

$$0 \le r \le k-1 \quad \text{for } s = 2k;$$

$$C_r(x,u) = \sum_{j=0}^{n_r} \gamma_{rj}(x) \frac{\partial^j u}{\partial t^j}, \qquad 0 \le r \le k-1;$$

* **Note added in proof.** Recently it has become known to the author that in (^{10}) the existence of a generalized solution of problem (1), (4)–(6) was proved under the conditions: a) A_{ij} are constants; b) $B(u) \equiv 0$; c) $s < m$; d) either $s = 4s_1,\ m = 2m_1 + 1$, or $s = 4s_1 + 2,\ m = 2m_1$.

$\alpha_{ri}, \beta_{ri}, \gamma_{ri}$ are sufficiently smooth functions given on $\Gamma, \Omega_0, \Omega_T$, respectively; the operators A_r, B_r, C_r cover, in the sense of (4), the operator L; $\tilde{p}_r = p(1 - l_r/q - 1/2q)$; $\tilde{q}_r = q(1 - l_r/q - 1/2q)$; $\tilde{\tilde{q}}_r = q(1 - m_r/p - 1/2p)$; $\tilde{\tilde{q}}_r = q(1 - n_r/p - 1/2p)$, $p \geq \max(s, m_r, n_r)$, $q \geq \max(2m, l_r)$. In the parabolic case, $s = 1$, Theorem 2 was formulated in [5].

Consider the special case of equation (1)

$$L_0(u) \equiv (-1)^{[s/2]+1} \frac{\partial^s u}{\partial t^s} - (-\Delta)^m u = f(x,t). \tag{9}$$

Theorem 3. The problem (9), (4)–(6) is uniquely solvable in $W^{(s,2m)}(Q)$ for arbitrary $f(x,t) \in \mathcal{L}_2(Q)$; $\varphi_r(x,t) \in W^{(2m-r-1/2s)2m,(2m-r-1/2)}(\Gamma)$, $\psi_r \in W^{2m(1-r/s-1/2s)}(\Omega_0)$; $\chi_r \in W^{2m(1-r/s-1/2s)}(\Omega_T)$.

To prove Theorem 3, again by means of the substitution $u(x,t) = w(x,t) - z(x,t)$, we reduce the problem (9), (4)–(6) to the problem (9), (4_0)–(6_0) with right-hand side in (9) $F(x,t) = f(x,t) - L_0(z)$. The latter problem is solved as follows: first a generalized solution $w(x,t) \in \overset{\circ}{W}{}^{[s/2],m}(Q)$ is constructed, satisfying the integral identity

$$\left[\frac{\partial^{[s/2]} w}{\partial t^{[s/2]}}, \frac{\partial^{[s/2]+1} v}{\partial t^{[s/2]+1}}\right] - \sum_{|i|=m} [D^i w, D^i v] = [f, v]. \tag{10}$$

The brackets $[\,,\,]$ denote integration over Q, and the function $v(x,t)$ is any function from $\overset{\circ}{W}{}^{[s/2]+1,m}(Q)$ satisfying the boundary conditions (4_0) and

$$v\Big|_{\substack{t=0\\t=T}} = \cdots = \frac{\partial^{k-1} v}{\partial t^{k-1}}\Big|_{\substack{t=0\\t=T}} = \frac{\partial^k v}{\partial t^k}\Big|_{t=T} = 0.$$

Next, uniqueness of the generalized solution in $\overset{\circ}{W}{}^{[s/2],m}(Q)$ is proved, from which one can obtain the estimate

$$\|w\|^2_{\overset{\circ}{W}{}^{[s/2],m}(Q)} \leq C_3^2 \|f\|^2_{L_2(Q)}, \tag{11}$$

where C_3 does not depend on $f(x,t)$. After this it is proved that the generalized solution is in fact a smooth solution from $W^{s,2m}(Q)$ of equation (9). The proof of the smoothness of the solution inside the domain Q and near those portions of its boundary which do not include some neighborhood of the set $(\Gamma \cap (t=0)) \cup (\Gamma \cap (t=T))$ is carried out with the aid of a partition of unity and Hörmander' s theorems. (We note that equation (1), as can be proved, is hypoelliptic.)

The proof of smoothness near $t=0$ $(t=T)$ (and hence also near the manifold $(\Gamma \cap (t=0)) \cup (\Gamma \cap (t=T)))$ is carried out in the following way: in (10), as the function $v(x,t)$ one substitutes

$$\rho(t) G_{N,\delta}(x,\xi,t,\tau) = \rho(t) \sum_{r=1}^{N} V_r(x) V_r(\xi) \int_0^{\infty} M_r(t,t_1) \xi_\delta(t-t_1)\, dt_1, \quad \delta > 0,\ N > l; \tag{12}$$

where $V_r(x)$ is the r-th eigenfunction of the operator $(-\Delta)^m$ in the domain Ω_0, corresponding to the eigenvalue λ_r, under the boundary conditions $V_r|_\gamma = \ldots$

$$\ldots = D^{m-1} V_r|_\gamma = 0;$$

$M_r(t,t_1)$ is the Green' s function of the operator

$$(-1)^{[s/2]} \frac{\partial^s}{\partial t^s} - \lambda_r,$$

satisfying the conditions

$$M_r(t,t_1)\big|_{t=0} = \cdots = \frac{d^{[s/2]-1}}{dt^{[s/2]-1}} M_r(t,t_1)\big|_{t=0} = 0, \qquad M_r\big|_{t=\infty} = 0,$$

$$\xi_\delta(t_1 - \tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\theta(t_1-\tau)} e^{-\delta\theta^2}\, d\theta, \qquad \delta > 0;$$

$\rho(t)$ is a function from $C^\infty([0,T])$; $|\rho(t)| \leqslant 1$; $\rho(t) = 1$ for $0 \leqslant t \leqslant T/3$; $\rho(t) = 0$ for $T/2 \leqslant t \leqslant T$.

From equality (10) we obtain

$$w_{N\delta}(\xi,\tau) = [tp, G_{N\delta}] + \sum_{p \leqslant k} B_p [(u\rho^{2k+1-p})_{tp}, G_{N\delta}] + \sum_{p \geqslant k+1} B_p \left[(u\rho^{2k+1-p})_{t^k} \frac{\partial^{p-k} G_{N\delta}}{\partial t^{p-k}} \right], \tag{13}$$

where $w_{N\delta}(\xi,\tau) = [w(x,t), L_0^*(x,t) G_{N\delta}(x,\xi,t,\tau)]$ are smooth functions, and $B_p,\ p = 0, \ldots, 2k$, are certain numbers.

It can be proved that for any $w(x,t) \in \mathcal{L}_2(Q)$ the sequence $w_{N\delta}(\xi,\tau)$ as $N \to \infty$ and $\delta \to 0$ converges in $\mathcal{L}_2(Q)$ to $w(\xi,\tau)$.

Using (13) and (11), one can prove

$$\left\| \frac{\partial^{[s/2]+1} w_{N\delta}}{\partial \tau^{[s/2]+1}} \right\|^2_{\mathcal{L}_2(Q)} \leqslant C_4^2 \|f\|^2_{\mathcal{L}_2(Q)},$$

whence, in turn, follows the existence of the $([s/2]+1)$-st derivative with respect to τ in $\mathcal{L}_2(Q \cap (0 \leqslant t \leqslant T/3))$ of $w(\xi, \tau)$. Analogously, by means of a suitably chosen $\widetilde{G}_{N\delta}(x, \xi, t, \tau)$, the existence of the $([s/2]+1)$-st derivative in $\mathcal{L}_2(Q \cap (2T/3 \leqslant t \leqslant T))$ is proved, and hence, by virtue of (6), also in $\mathcal{L}_2(Q)$. Then the existence in $\mathcal{L}_2(Q)$ of the $([s/2]+2)$-nd derivative of $w(\xi, \tau)$ with respect to τ, etc., up to the derivative with respect to τ of order s of $w(\xi, \tau)$, is established. In this case the estimate

$$\|w_{t^s}\|^2_{\mathcal{L}_2(Q)} \leqslant C_5^2 \|f\|^2_{\mathcal{L}_2(Q)}$$

holds.

From this estimate, equation (9) for $w(x,t)$, and a known theorem from (8), the proof of Theorem 3 follows.

By the method of continuation with respect to a parameter (for the parabolic equation (1) ($s = 1$) this method is presented in (9)), with the aid of Theorem 1 on an a priori estimate, the following theorems are proved.

Theorem 4. For problem (1), (4)–(6) and its adjoint problem (1^*), (4^*)–(6^*), all Fredholm theorems are valid.

In particular, for the existence of a unique solution of (1), (4)–(6) it is necessary and sufficient that problem (1_0), (4_0)–(6_0) have only the trivial solution.

Theorem 5. If for every smooth finite function $v(x,t)$ in Q the inequality

$$[(-1)^m \sum D^i A^{ij}(x,t) D^{jv} - B(v),\, v] \geqslant C_6^2 [v, v]$$

holds with a constant C_6 independent of v, then problem (1), (4)–(6) is uniquely solvable in $W^{(s,2m)}(Q)$ for all right-hand sides and boundary functions under consideration.

Theorem 6. Problem (1), (4)–(6) is uniquely solvable in $W^{s,2m}(Q)$, if the height T of the cylinder Q is sufficiently small.

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Note: Figure translations are in progress. See original paper for figures.

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