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Abstract

Full Text

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ON SECOND-ORDER DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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We consider the problem

$$v'' + A(t)v' + B(t)v = f(t) \quad (0 \le t \le T); \qquad v(0) = v_0, \quad v'(0) = v_0'. \tag{1}$$

in a Banach space E. We shall call a function $v(t)$ a solution of problem (1) if, for all t in $[0, T]$, it satisfies equation (1) and the functions $v''(t)$, $A(t)v'(t)$, and $B(t)v(t)$ are continuous on $[0, T]$.

In this note we give a theorem on the existence of a solution of problem (1), as well as of some nonlinear problems. An equation with a small parameter $\varepsilon > 0$ at the derivative $v''(t)$ is considered, and it is shown that the solution of this problem tends to the solution of a degenerate first-order equation. We note that the stimulus for writing this paper was the work (^{1}), in which problem (1) with a small parameter and constant operators A and B was considered. The method presented below for investigating problem (1), etc., differs from the method developed in (^{1}). The latter method, apparently, does not generalize to equations with variable operators.

1. In the work (^{2}) the problem

$$v' + A(t)v = 0 \quad (0 \le s \le t \le T); \qquad v(s) = v_0 \tag{2}$$

was studied. The solution of this problem is written in the form $U(t, s)v_0$. The operator $U(t, s)$ makes it possible to determine the solution of the problem

$$v' + A(t)v = f(t) \quad (0 \le t \le T); \qquad v(0) = v_0 \tag{3}$$

by the formula

$$v(t) = U(t, 0)v_0 + \int_0^t U(t, s)f(s)\, ds. \tag{4}$$

Let now $v(t)$ be a solution of problem (1). Integrating (1) from 0 to t, performing integration by parts and using formula (4), we arrive at the relation

$$v(t) = U(t,0)v_0 + \int_0^t U(t,s)\left\{v_0' + A(0)v_0 + \int_0^s f(\tau)d\tau + \int_0^s [A'(\tau) - B(\tau)]v(\tau)d\tau\right\} ds. \tag{5}$$

Finally, making the substitution $x(t) = A(t)v(t)$ and applying formula (2.45) from (^{3}), we obtain

$$\begin{aligned} x(t) = {} & \left[I + \int_0^t W(t,\tau)A'(\tau)A^{-1}(\tau)d\tau\right] A(0)v_0 + \left[I - W(t,0) + \int_0^t W(t,\tau)A'(\tau)A^{-1}(\tau)d\tau\right] v_0' \\ & + \int_0^t \left[I - W(t,\tau) + \int_\tau^t W(t,s)A'(s)A^{-1}(s)ds\right] f(\tau)d\tau \\ & + \int_0^t \left[I - W(t,\tau) + \int_0^t W(t,s)A'(s)A^{-1}(s)ds\right] [A'(\tau) - B(\tau)]A^{-1}(\tau)x(\tau)d\tau, \end{aligned} \tag{6}$$

where $W(t,\tau) = A(t)U(t,\tau)A^{-1}(\tau)$.

Thus it is proved:

Theorem 1. *Suppose that the operator $-A(t)$, for each $t \in [0,T]$, is the infinitesimal generator of a strongly continuous semigroup* $\exp\{-\tau A(t)\}$

($\tau \geqslant 0$), whose norm satisfies the inequality

$$\|\exp\{-\tau A(t)\}\| \leqslant \exp\{-\delta\tau\} \quad (\delta > 0). \tag{7}$$

Suppose that the operator $A(t)$ has a domain of definition D independent of t, and that the operator-function $A(t)A^{-1}(0)$ is twice strongly continuously differentiable. Suppose that the operator-function $B(t)A^{-1}(0)$ is once strongly continuously differentiable. Suppose that the function $f(t)$ is continuously differentiable. Finally, suppose that v_0 and v_0' belong to D. Then problem (1) has one and only one solution.

In proving the smoothness of solutions of equation (8), the following is used.

Lemma 1. Suppose the inequality

$$\max_{0\leqslant t\leqslant T} \left\|A'(t)A^{-2}(t)\right\| < 1 \tag{8}$$

holds. Then the identity

$$\begin{aligned}\int_0^t W(t,s)f(s)\,ds = A^{-1}(t)\left[I - A'(t)A^{-2}(t)\right]^{-1} f(t)& \\ - W(t,0)A^{-1}(0)\left[I - A'(0)A^{-2}(0)\right]^{-1} f(0)& \\ + \int_0^t W(t,s)A^{-1}(s)A'(s)A^{-1}(s)\left[I - A'(s)A^{-2}(s)\right]^{-1} f(s)\,ds& \\ - \int_0^t W(t,s)A^{-1}(s)\left(\left[I - A'(s)A^{-2}(s)\right]^{-1}\right)' f(s)\,ds& \\ - \int_0^t W(t,s)A^{-1}(s)\left[I - A'(s)A^{-2}(s)\right]^{-1} f'(s)\,ds.&\end{aligned} \tag{9}$$

The requirement (8) imposes essentially no additional restrictions on the problem, since by means of the substitution $t = kt$ one can always ensure that it is satisfied.

2. Consider the nonlinear problem

$$v'' + A(t)v' = f(t, v, v') \quad (0 \leqslant t \leqslant T); \qquad v(0) = v_0, \quad v'(0) = v_0'. \tag{10}$$

Theorem 2. Suppose that the operator $A(t)$ and the elements v_0 and v_0' satisfy the conditions of Theorem 1. Suppose that the operator $f(t, A^{-1}(0)v, w)$ is differentiable with respect to $t \in [0, T]$ and $v, w \in E$ (the derivatives with respect to v, w are Fréchet derivatives). Suppose that the derivatives $f_t(t, A^{-1}(0)v, w)$, $f_v(t, A^{-1}(0)v, w)$, and $f_w(t, A^{-1}(0)v, w)$ are continuous in the aggregate of the variables and bounded when v and w belong to a bounded set in E. Then problem (10) has one and only one solution, defined on some segment $[0, t_0] \subset [0, T]$.

3. Consider the problem with a small parameter

$$\varepsilon v'' + A(t)v' = f(t, v) \quad (0 \leqslant t \leqslant T); \qquad v(0) = v_0, \quad v'(0) = v_0' \quad (0 < \varepsilon \leqslant 1) \tag{11}$$

and the corresponding degenerate problem ($\varepsilon = 0$)

$$A(t)v' = f(t, v) \quad (0 \leqslant t \leqslant T); \qquad v(0) = v_0. \tag{12}$$

Theorem 3. Suppose the conditions of Theorem 2 are fulfilled. Then problem (11), for every $\varepsilon \in (0, 1]$, has a solution $v_\varepsilon(t)$, defined on some segment $[0, t_0] \subset [0, T]$ independent of ε. As $\varepsilon \to 0$, the function $A(t)v_\varepsilon(t)$ converges uniformly on $[0, T]$ to the function $A(t)v_0(t)$, where $v_0(t)$ is the solution of problem (12). The function $A(t)v_\varepsilon'(t)$ is uniformly bounded with respect to ε on $[0, T]$, and for $t > 0$ it converges to the function $A(t)v_0'(t)$ as $\varepsilon \to 0$. Finally, the residual $A(t)v_\varepsilon'(t) - f[t, v_\varepsilon'(t)]$ is uniformly bounded with respect to ε on $[0, T]$, and for $t > 0$ tends to zero as $\varepsilon \to 0$.

The proof of Theorem 3 is based on the following lemmas.

Lemma 2. *Let the operator-function $A(t)A^{-1}(0)$ be once strongly continuously differentiable. Then*

$$\|U_\varepsilon(t,s)\| \le \exp\left\{-\frac{t-s}{\varepsilon}\delta\right\}, \tag{13}$$

$$\|W_\varepsilon(t,s)\| \le C\exp\left\{-\frac{t-s}{\varepsilon}\delta\right\}. \tag{14}$$

Let the operator-function $A(t)A^{-1}(0)$ be twice strongly continuously differentiable. Then

$$\|A(t)W_\varepsilon(t,s)A^{-1}(s)\| \le C\exp\left\{-\frac{t-s}{\varepsilon}\delta\right\}. \tag{15}$$

Here $U_\varepsilon(t,s)$ and $W_\varepsilon(t,s)$ are operator-functions constructed from the operator-function $\frac{1}{\varepsilon}A(t)$ in the same way as the operator-functions $U(t,s)$ and $W(t,s)$ are constructed from the operator-function $A(t)$.

Lemma 3. *Let the operator-function $A(t)A^{-1}(0)$ be once strongly continuously differentiable, and let the function $f(t)$ be strongly continuously differentiable. Then*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}A(t)\int_0^t U_\varepsilon(t,s)f(s)\,ds = f(t). \tag{16}$$

If, moreover, $A(t)A^{-1}(0)$ is twice strongly continuously differentiable, then

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}A(t)\int_0^t W_\varepsilon(t,s)f(s)\,ds = f(t). \tag{17}$$

4. Problem (1) was investigated by us under the assumption that the operator B is subordinate to the operator A. We shall show that in some cases this requirement can be weakened. Let $v(t)$ be a solution of problem (1). Then it is easy to see that the identity

$$\left[A(t)\int_0^t v'\,ds\right]' + A_1(t)\left[A(t)\int_0^t v'\,ds\right] = f(t) - B(t)v_0 - v'', \tag{18}$$

holds, where $A_1(t) = B(t)A^{-1}(t) - A'(t)A^{-1}(t)$. Let $U_1(t,s)$ be the operator-function with whose aid the solution of the problem $v' + A_1(t)v = 0$ $(0 \le s \le t \le T)$, $v(s) = v_0$, is written in the form $U_1(t,s)v_0$. Then from (19) it follows that

$$A(t)\int_0^t v'\,ds = \int_0^t U_1(t,s)[f(s) - B(s)v_0]\,ds - \int_0^t U_1(t,s)v''\,ds. \tag{19}$$

Integrating by parts in the last integral, we find:

$$\left[\int_0^t v'\,ds\right]' + A(t)\left[\int_0^t v'\,ds\right] =$$

$$= U_1(t,0)v_0' + \int_0^t U_1(t,s)A_1(s)v'\,ds + \int_0^t U_1(t,s)[f(s) - B(s)v_0]\,ds. \tag{20}$$

Hence it follows that

$$\int_0^t v'\,ds = \int_0^t U(t,\tau)\left\{U_1(\tau,0)v_0' + \int_0^\tau U_1(\tau,s)[f(s) - B(s)v_0]\,ds\right\}d\tau +$$

$$+ \int_0^t U(t,\tau)\left\{\int_0^\tau U_1(\tau,s)A_1(s)v'\,ds\right\}d\tau. \tag{21}$$

Using formula (2.46) from $^{(3)}$, we obtain:

$$\begin{aligned}
v'(t) = \Bigg[& W(t,0) - \int_0^t W(t,s)A'(s)A^{-1}(s)U_1(s,0)\,ds \\
& + \int_0^t W(t,s)\frac{\partial}{\partial s}U_1(s,0)\,ds\Bigg] v'_0 - \int_0^t W(t,\tau)A'(\tau)A^{-1}(\tau)\left\{\int_0^\tau U_1(\tau,s)[f(s)- \right.\\
& -B(s)v_0]\,ds\}\,d\tau + \int_0^t \Bigg[W\left(t,\frac{t+s}{2}\right)U_1\left(\frac{t+s}{2},s\right) + \int_{\frac{t+s}{2}}^t W(t,\tau)\frac{\partial}{\partial\tau}U_1(\tau,s)\,d\tau - \\
& - \int_s^{\frac{t+s}{2}} \frac{\partial}{\partial\tau}W(t,\tau)U_1(\tau,s)\,d\tau\Bigg][f(s)-B(s)v_0]\,ds - \int_0^t W(t,\tau)A'(\tau)A^{-1}(\tau)\times \\
& \times\left\{\int_0^\tau U_1(\tau,s)A_1(s)v'(s)\,ds\right\}d\tau + \int_0^t \Bigg[W\left(t,\frac{t+s}{2}\right)U_1\left(\frac{t+s}{2},s\right) + \\
& + \int_{\frac{t+s}{2}}^t W(t,\tau)\frac{\partial}{\partial\tau}U_1(\tau,s)\,d\tau - \int_s^{\frac{t+s}{2}} \frac{\partial}{\partial\tau}W(t,\tau)U_1(\tau,s)\,d\tau\Bigg]A_1(s)v'(s)\,ds.
\end{aligned} \tag{22}$$

Thus, for example, the following is proved:

Theorem 4. Let, for every $t \in [0,T]$ and any λ with $\operatorname{Re}\lambda \geq 0$, the operator $A(t) + \lambda I$ have a bounded inverse, whose norm satisfies the inequality*

$$\left\|[A(t)+\lambda I]^{-1}\right\| \leq C[|\lambda|+1]^{-1}. \tag{23}$$

Suppose the operator $A(t)$ has a dense domain of definition D in E, independent of t; suppose the operator-function $A(t)A^{-1}(0)$ is continuously differentiable and its derivative $A'(t)A^{-1}(0)$ satisfies some Hölder condition. Suppose the operator $A_1(t)$ satisfies condition (23), has a domain of definition D_1 independent of t, and suppose the operator-function $A_1(t)A_1^{-1}(0)$ satisfies some Hölder condition. Suppose the operator-function $A_1(t)A^{-\rho}(0)$ satisfies a Hölder condition for some $\rho \in [0,1)$. Suppose the functions $B(t)v_0$ and $f(t)$ satisfy some Hölder condition. Finally, suppose $v_0 \in D[A^{\rho_1}(0)]$ for some $\rho_1 > \rho$. Then problem (1) has one and only one solution.

By a solution here is meant a function $v(t)$, continuously differentiable once on $[0,T]$, satisfying the initial conditions (1), and, for $t > 0$, equation (1), such that the functions $v''(t)$ and $A(t)v'(t)$ are continuous for $t > 0$, while the functions $\|v''(t)\|$ and $\|A(t)v'(t)\|$ are summable on $[0,T]$.

Similarly, one can consider the nonlinear problem

$$v'' + A(t)v' + B(t,v,v')v = f(t,v,v'); \qquad v(0) = v_0, \quad v'(0) = v'_0 \tag{24}$$

and the problem with a small parameter at the highest derivative.

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[*] For operators satisfying (23), all fractional powers are defined (4).

Note: Figure translations are in progress. See original paper for figures.