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Abstract

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ON THE SPECTRAL THEORY OF NON-SELF-ADJOINT DIFFERENTIAL OPERATORS

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1. In note ([1]) the author considered certain spectral properties of a non-self-adjoint ordinary differential operator. The present note is devoted to a further study of the spectral properties of this operator, as well as of the corresponding operator in partial derivatives.

In the Hilbert space $L_2(0,\infty)$ consider the differential operator

$$ly = -y'' + q(x)y, \qquad y(0) - hy'(0) = 0, \tag{1}$$

where the continuous function $q(x)$ and the number h, generally speaking, are complex. It is known that under the condition

$$\int_0^\infty |q(x)|\, x\, dx < \infty \tag{2}$$

the equation $ly = k^2 y$ for each k from the half-plane $\operatorname{Im} k \geqslant 0$ has a unique solution $f(x,k)$, satisfying, as $x \to \infty$, the condition $f(x,k)\exp(-ikx) \to 1$, and moreover $f(x,k)$ and $f'_x(x,k)$ are regular in the upper half-plane $\operatorname{Im} k > 0$ and continuous up to the real axis.

The point $\lambda_0 = k_0^2$ is called a singular point of the operator (1) (see ([1])) if k_0 is a zero of the function $D(k) = f(0,k) - hf'_x(0,k)$. The Weyl function ([2]) of the operator (1) is expressed in terms of $f'_x(0,k)$ and $f(0,k)$ by the formula

$$m(\lambda) = [f'_x(0,k) + hf(0,k)]D^{-1}(k) \qquad (\lambda = k^2).$$

The set of singular points $\lambda \in [0,\infty)$ coincides with the set of eigenvalues. We shall denote the set of singular points $\lambda \in [0,\infty)$ by E.

2. We shall assign the function $q(x)$ to the class S_n, $n = 0,1,2,\ldots$, if $q(x)$ is continuous on the interval $[0,\infty)$ and

$$\int_0^\infty |q(x)|\, x^{n+1}\, dx < \infty.$$

The function $q(x) \in S_\infty$ if $q(x) \in S_n$ for every n. If $q(x) \in S_n$, then, as is known, the function $D(k)$ has derivatives up to order n, continuous up to the real axis, and hence one may speak of the multiplicity of a singular point in the case when the notion of multiplicity has meaning for the corresponding zero of the function $D(k)$.

We shall call two bounded sets on the real axis similar if some intervals containing them can be transformed into one another by a one-to-one continuously differentiable mapping in such a way that, under this mapping, one set passes into the other.

By the methods of note ([1]), with the aid of a theorem of Carleson ([3]), the following assertions are established.

Theorem 1. *If $q(x) \in S_n$, $0 < n < \infty$, then the set E has linear measure zero, is bounded, closed, and satisfies the condition*

$$\sum l_\nu \log l_\nu > -\infty. \tag{3}$$

Here l_ν are the lengths of the ν-th intervals of contiguity to the set E, and the summation extends over all bounded intervals of contiguity to the set E.

Theorem 2. For every bounded set E_0 satisfying the conditions of Theorem 1, and for any integer $n > 0$, there exists an operator of the form (1) with an infinitely differentiable potential $q(x) \in S_n$ such that the set E of all its real exceptional points coincides with the set of accumulation points of eigenvalues and is similar to E_0.

Theorem 3. Let $q(x) \in S_\infty$. Then the set E_1 of exceptional points of finite multiplicity satisfies the condition

$$\int_0 \log T(t)\, d\varphi_{E_1}(t) > -\infty,$$

where

$$T(t) = \inf_s [(s+1)^{-1} C_{s+1} + C_s] t^s [s!]^{-1},$$

and $\varphi_{E_1}(t)$ is the Lebesgue measure of the t-neighborhood of the set E_1.

Corollary. If the potential $q(x)$ is such that

$$\int_0 \log T(t)\, dt = -\infty, \tag{4}$$

then the number of exceptional points of the operator (1) and their total multiplicity are finite.

What has been said is a strengthening of Theorem 1 of note (1), since condition (4) is weaker than the condition of quasianalyticity of the function $D(k)$. Condition (4) is satisfied, for example, for functions $q(x)$ satisfying the inequality

$$|q(x)| \leq C \exp(-\varepsilon\sqrt{x}), \qquad \varepsilon > 0.$$

In this connection, note that the condition

$$|q(x)| \leq C \exp(-bx^\alpha)$$

does not ensure the finiteness of the number of eigenvalues for any $\alpha < 0.5$.

Analogous assertions are true in the three-dimensional case. In the Hilbert space $L_2(R_3)$ consider the operator

$$-\Delta u + qu, \tag{5}$$

where $q(x)$ is a continuous complex-valued function. The following is valid.

Theorem 4. Let

$$\sup_x |x|^k |q(x)| < \infty$$

for every $k > 0$, and

$$q_k = \sup_x |x|^k (1 + |x|^4) |q(x)| + \sup_x |x|^k (1 + |x|^4) |q(x)|^{1/2}, \qquad k = 0, 1, 2, \dots,$$

$$Q_n = \sup \prod_i q_{n_i} [n_i!]^{-1},$$

where the supremum is taken over all collections $\{n_i\}$, $\sum n_i = n$, $n_i \geq 1$.

Then the set $\widetilde{E}$ of accumulation points of eigenvalues of the operator (5) satisfies the condition

$$\int_0 \log T(t)\, d\varphi_{\widetilde{E}}(t) > -\infty.$$

Here

$$T(t) = \inf_n t^n Q_n;$$

$\varphi_{\widetilde{E}}(t)$ is the Lebesgue measure of the t-neighborhood of the set $\widetilde{E}$.

Corollary. If

$$\int_0 \log T(t)\, dt = -\infty,$$

then the number of eigenvalues of the operator (5) is finite. In particular, this is so if

$$|q(x)| < C \exp(-\varepsilon \sqrt{|x|}), \qquad \varepsilon > 0.$$

Remark. If

$$\sup_x |x|^k |q(x)| < \infty$$

for some $k > 4$, then condition (3) is fulfilled for the system of intervals of contiguity of the set $\widetilde{E}$.

3. In ([1]) it was shown that the series in eigenfunctions of the discrete spectrum of the operator (1), generally speaking, diverges. However, one can indicate a method of summing this series, consisting in the fact that the terms of the series are "corrected" by combinations of "eigen" and "associated" functions corresponding to points of accumulation of the eigenvalues.

Let us consider, for simplicity of notation, the operator (1) with the boundary condition $y'(0) = 0$. In addition, we shall assume that the set of eigenvalues has a single limiting point λ_0. For the boundary condition chosen by us, the Weyl function is $m(\lambda) = f(0,k)/f'_x(0,k)$, $k^2 = \lambda$.

Let $p_s[(\lambda - \lambda_s)^{-1}]$ be the principal part of the Weyl function $m(\lambda)$ at the pole λ_s. In the usual way (see ([4])) we construct a function $m_d(\lambda)$, regular everywhere except for the poles λ_s and the point λ_0, and having at the poles the same principal parts as $m(\lambda)$:

$$m_d(\lambda) = \sum_{s=1}^{\infty} \left\{ p_s[(\lambda - \lambda_s)^{-1}] + q_s[(\lambda - \lambda_0)^{-1}] \right\}.$$

Under condition (2) there is an N such that on the interval $[N, \infty)$ there will be no singular points of the operator (1). The function $m(\lambda)$ can be represented as the sum of three terms

$$m(\lambda) = m_d(\lambda) + m_c^0(\lambda) + m_c^N(\lambda), \tag{6}$$

where $m_c^N(\lambda)$ is a function regular in the λ-plane with the cut $[N, \infty)$, and $m_c^0(\lambda)$ is regular outside the interval $[0, N]$ and bounded at infinity.

Let $f(x)$ and $g(x)$ be finite functions from the domain of definition of the operator under consideration, and let $\varphi(x, \lambda)$ be the solution of the equation $ly = \lambda y$ satisfying the initial conditions $\varphi'(0, \lambda) = 0,\ \varphi(0, \lambda) = 1$. Further, let γ be any contour enclosing the spectrum of the operator (1), and

$$\tilde{f}(\lambda) = \int_0^\infty f(x)\varphi(x, \lambda)\, dx.$$

Then the "Parseval equality" holds (see ([2]))

$$(f, g) = (2\pi i)^{-1} \oint_\gamma \tilde{f}(\lambda)\, \tilde{\tilde{g}}(\lambda)\, m(\lambda)\, d\lambda. \tag{7}$$

Using the decomposition (6), we rewrite (7) in the form

$$\begin{aligned} 2\pi i(f, g) = \oint_{\gamma_d} \tilde{f}(\lambda)\tilde{\tilde{g}}(\lambda) m_d(\lambda) + \int_{\gamma_0} \tilde{f}(\lambda)\tilde{\tilde{g}}(\lambda) m_c^0(\lambda)\, d\lambda \\ + \int_{\gamma_N} \tilde{f}(\lambda)\tilde{\tilde{g}}(\lambda) m_c^N(\lambda)\, d\lambda = J_d + J_0 + J_N. \end{aligned} \tag{8}$$

Here γ_d is a contour enclosing the discrete spectrum of the operator (1) and its limiting points; γ_0 is a contour enclosing the interval $[0, N]$; γ_N is a contour enclosing the half-axis $[N, \infty)$. The last integral, in view of the absence of singular points on $[N, \infty)$, can be rewritten in the form

$$J_N = \int_N^\infty [m(\lambda - i0) - m(\lambda + i0)]\tilde{f}(\lambda)\tilde{\tilde{g}}(\lambda)\, d\lambda. \tag{9}$$

The integral just written converges absolutely and essentially does not differ from the corresponding term in Parseval' s equality for the self-adjoint case.

Let us consider the term corresponding to the discrete spectrum:

$$J_d = \oint_{\gamma_d} \tilde{f}(\lambda)\tilde{\tilde{g}}(\lambda) m_d(\lambda)\, d\lambda =$$

$$= \sum_{s=1}^\infty \left\{ \tilde{p}_s \left(\frac{d}{d\lambda}\right) [\tilde{f}(\lambda)\tilde{\tilde{g}}(\lambda)]\Big|_{\lambda=\lambda_s} + \tilde{q}_s \left(\frac{d}{d\lambda}\right) [\tilde{f}(\lambda)\tilde{\tilde{g}}(\lambda)]\Big|_{\lambda=\lambda_0} \right\}. \tag{10}$$

The polynomials $\tilde{p}_s$ are related to the polynomials p_s by the equality

$$2\pi i\,\tilde{p}(\omega) = \sum_{k>0} p^{(k)}(0)\,[k!(k-1)!]^{-1}\,\omega^{k-1}.$$

The polynomials $\tilde{q}_s$ are constructed analogously from q_s. The series (10) converges absolutely for arbitrary finite functions $f(x)$ and $g(x)$. If λ_s is an eigenvalue of rank t_s, then the functions $\varphi_\lambda^{(t)}(x,\lambda_s)$, $t = 0, 1, \dots, t_s - 1$, belong to $L_2(0,\infty)$ and form a chain of principal functions corresponding to the eigenvalue λ_s. In this case the operator on $f(x)$

$$\tilde{p}_s\left(\frac{d}{d\lambda}\right)\left[\tilde{f}(\lambda)\varphi(x,\lambda)\right]\Big|_{\lambda=\lambda_s}$$

is a (nonorthogonal) projector onto the root subspace corresponding to the eigenvalue λ_s, and for any finite function $f(x)$ the equality

$$(l-\lambda_s I)^{t+1}\left\{\left(\frac{d}{d\lambda}\right)^t\left[\tilde{f}(\lambda)\varphi(x,\lambda)\right]\Big|_{\lambda=\lambda_s}\right\} = 0. \tag{11}$$

The functions $\varphi_\lambda^{(t)}(x,\lambda_0)$, $t = 0, 1, \dots$, no longer belong to $L_2(0,\infty)$; however, an equality of the form (11) holds for any finite function $f(x)$. The function $\varphi_\lambda^{(t)}(x,\lambda_0)$ plays the role of an associated function of order t at the singular point λ_0, lying in the continuous spectrum. We note that in the self-adjoint case the Weyl function satisfies the condition $\operatorname{Im} m(\lambda)\cdot\operatorname{Im}\lambda < 0$, and therefore (see ([5])) the series composed of the principal parts of $m(\lambda)$ converges absolutely and uniformly in every strictly interior subdomain of regularity of $m(\lambda)$.

The middle term of formula (6), with the aid of a theorem of V. P. Khavin ([6]), is transformed to the form

$$J_0 = \sum_{n=0}^{\infty}[n!]^{-1}\int_0^N f_n(\lambda)\left(\frac{d}{d\lambda}\right)^n[\tilde{f}(\lambda)\tilde{g}(\lambda)]\,d\lambda, \tag{12}$$

where $f_n(\lambda) \subset L_2(0,N)$ and $(\|f_n\|)^{1/n} \to 0$ as $n \to \infty$. The series (12) converges absolutely. We obtain the final form of Parseval' s equality by combining the representations (9), (10), (12).

We note that $\tilde{q}_s$ and $f_n(\lambda)$ are determined by $m(\lambda)$ nonuniquely. In particular, for any $\varepsilon > 0$ one can specify a collection $\{f_n(\lambda)\}$ such that all $f_n(\lambda)$, except $f_0(\lambda)$, are equal to zero outside the ε-neighborhood of the set E. In this sense one may say that the generalized spectral measure is absolutely continuous on intervals adjacent to the set of singular points. The following assertion is true:

Theorem 5. *If on the segment* $[a, b] \subset [0, \infty)$ *the multiplicity of singular points does not exceed* t_0, *then there exists a collection* $\{f_n(\lambda)\}$ *such that* $f_n(\lambda) = 0$ *for* $\lambda \in [a, b]$ *for all* $n > t_0 + 1$, *while* $f_{t_0+1}(\lambda)$ *is bounded on* $[a, b]$.

The case in which eigenvalues accumulate on an arbitrary closed set of measure zero reduces to the one considered, since the set of eigenvalues can be represented as the sum of a countable number of disjoint sets, each of which has a single limit point on the real half-axis $[0, \infty)$.

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Note: Figure translations are in progress. See original paper for figures.