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ON THE DIFFERENTIABILITY OF FOURIER SERIES IN GENERALIZED SPHERICAL FUNCTIONS

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Abstract

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MATHEMATICS

S. S. LITVINKOV

ON THE DIFFERENTIABILITY OF FOURIER SERIES IN GENERALIZED SPHERICAL FUNCTIONS

(Presented by Academician V. I. Smirnov on 23 I 1962)

1°. In the article ([1]) we considered the question of expanding vector functions on the surface of a sphere into uniformly and pointwise convergent series in generalized spherical functions ([2]). In the present note, we study the question of the differentiability of a Fourier series in generalized spherical functions and obtain theorems analogous to the theorems of S. G. Mikhlin for Fourier series in spherical functions ([3]).

Consider functions $u(\varphi_1, \theta, \varphi_2)$, periodic with period 2π in φ_1 and φ_2, in the parallelepiped $\sigma = (0 \leqslant \varphi_1 \leqslant 2\pi;\ 0 \leqslant \theta \leqslant \pi;\ 0 \leqslant \varphi_2 \leqslant 2\pi)$. By $L_2(\sigma)$ we denote the Hilbert space with scalar product

$$(u, v) = \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{\pi} \int_0^{2\pi} u(\varphi_1, \theta, \varphi_2) \cdot \overline{v_1(\varphi_1, \theta, \varphi_2)} \sin\theta \, d\varphi_1 \, d\theta \, d\varphi_2.$$

By B_1, B_2, and B_3 we denote the operators:

$$B_1 = e^{i\varphi_2} \left(\operatorname{ctg}\theta \frac{\partial}{\partial\varphi_2} - \frac{1}{\sin\theta} \frac{\partial}{\partial\varphi_1} + i \frac{\partial}{\partial\theta} \right);$$

$$B_2 = e^{i\varphi_2} \left(-\operatorname{ctg}\theta \frac{\partial}{\partial\varphi_2} + \frac{1}{\sin\theta} \frac{\partial}{\partial\varphi_1} + i \frac{\partial}{\partial\theta} \right), \qquad B_3 = i \frac{\partial}{\partial\varphi_2}.$$

On the set of functions $u(\varphi, \theta, \varphi_2)$ for which $B_1^{k_1} B_2^{k_2} B_3^{k_3} u$ $(k_1 + k_2 + k_3 = 1, 2, 3, \dots, r)$ are continuous, introduce the norm

$$\|u(\varphi_1, \theta, \varphi_2)\|_{W_2^r(B,\sigma)} = \sum_{k=0}^{r} \sum_{k_1+k_2+k_3=k} \left\| B_1^{k_1} B_2^{k_2} B_3^{k_3} u \right\|_{L_2(\sigma)}.$$

The completion of this set of functions with respect to the norm introduced will be called the space $W_2^r(B, \sigma)$.

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From the commutation relations for the operators B_1, B_2, B_3 (^{2}) it follows that every expression of the form

$$B_{i_1}^{\gamma_1} B_{i_2}^{\gamma_2} \cdots B_{i_m}^{\gamma_m} u \quad (i_1, i_2, \dots, i_m = 1, 2, 3),$$

where $\gamma_1 + \gamma_2 + \cdots + \gamma_m = r$, can be expressed through a linear combination of $B_1^{k_1} B_2^{k_2} B_3^{k_3} u$ $(k_1 + k_2 + k_3 \leqslant r)$.

In $W_2^r(B, \sigma)$ introduce the scalar product

$$[u, v] = \frac{1}{8\pi^2} \int_\sigma \sum_{k=0}^{r} \sum_{k_1+k_2+k_3=k} \left(B_1^{k_1} B_2^{k_2} B_3^{k_3} u\right) \left(\overline{B_1^{k_1} B_2^{k_2} B_3^{k_3} v}\right) d\sigma.$$

Here $d\sigma = \sin\theta \, d\varphi_1 \, d\theta \, d\varphi_2$. Thus, $W_2^r(B, \sigma)$ will be a Hilbert space.

The generalized spherical functions $T_{mn}^l(\varphi_1, \theta, \varphi_2)$ constitute a complete system of eigenfunctions of the self-adjoint operator

$$\Delta_2 = -\left[\frac{\partial^2}{\partial\theta^2} + \operatorname{ctg}\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi_1^2} - 2\cos\theta \frac{\partial^2}{\partial\varphi_1 \partial\varphi_2} + \frac{\partial^2}{\partial\varphi_2^2}\right)\right],$$

corresponding to the eigenvalues $l(l+1)$.

Definition. Every finite linear combination of generalized spherical functions will be called a **generalized spherical polynomial**.

Since the system of generalized spherical functions is complete in $L_2(\sigma)$, the set of generalized spherical polynomials is everywhere dense in $L_2(\sigma)$.

Lemma 1. *The set of generalized spherical polynomials is everywhere dense in* $W_2^r(B, \sigma)$.

Let $u \in W_2^r(B, \sigma)$ be orthogonal to all generalized spherical polynomials in $W_2^r(B, \sigma)$, i.e. $[u, v] = 0$, if v is a generalized spherical polynomial. Integrating by parts, we obtain

$$[u, v] = \frac{1}{8\pi^2} \int_\sigma \sum_{k=0}^{r} (-1)^k \sum_{k_1=k_2=k_3=k} u \, (B_3^{k_3} B_2^{k_2} B_1^{k_1} v) \, \overline{(B_1^{k_1} B_2^{k_2} B_3^{k_3} v)} \, d\sigma.$$

For generalized spherical functions,

$$\left[B_3^{k_3} B_2^{k_2} B_1^{k_1} \left(\overline{E_1^{k_1} B_2^{k_2} B_3^{k_3} T_{mn}^l}\right)\right] = (-1)^{k_1 k_2 k_3} (\lambda_{mn}^l)_{k_1+k_2+k_3} \overline{T_{mn}^l},$$

where $(\lambda_{mn}^l)_{k_1+k_2+k_3} \geq 0$ and $(\lambda_{mn}^l)_0 = 1$.

Denote

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$$\sum_{k=0}^{r} \sum_{k_1+k_2+k_3=k} (\lambda_{mn}^l)_{k_1+k_2+k_3} = (\gamma_{mn}^l)_r.$$

Then

$$\sum_{k=0}^{r}(-1)^k \sum_{k_1+k_2+k_3=k} \left[B_3^{k_3} B_2^{k_2} B_3^{k_3} \left(\overline{B_1^{k_1} B_2^{k_2} B_3^{k_3} T_{mn}^l}\right)\right] =$$

$$= \sum_{k=0}^{r} \sum_{k_2+k_2+k_3=k} (\lambda_{mn}^l)_{k_1+k_2+k_3} \overline{T_{mn}^l} = (\gamma_{mn}^l)_r \overline{T_{mn}^l},$$

where $(\gamma_{mn}^l)_r \geq 1$.

Using this identity, for every generalized spherical polynomial w one can construct such a polynomial v that

$$\sum_{k=0}^{r}(-1)^k \sum_{k_1+k_2+k_3=k} B_3^{k_3} B_2^{k_2} B_1^{k_1} \left(\overline{B_1^{k_1} B_2^{k_2} B_3^{k_3} v}\right) = w.$$

Then

$$[u, v] = \frac{1}{8\pi^2} \int_\sigma u \cdot \overline{w} \, d\sigma = (u, w).$$

But $[(u, w)] = 0$, and this means that $u = 0$, i.e. there does not exist a nonzero element $u \in W_2^r(B, \sigma)$ which would be orthogonal to the set of generalized spherical polynomials. Lemma 1 is proved.

2°. As is known (see (4)), the domain of definition of the self-adjoint operator $\Delta_2^{r/2}$ consists of those and only those functions

$$u = \sum_{l=0}^{\infty} \sum_{m,n=-l}^{l} C_{mn}^l \sqrt{2l+1}\, T_{mn}^l(\varphi_1, \theta, \varphi_2), \tag{1}$$

for which

$$\sum_{l=0}^{\infty} \sum_{m,n=-l}^{l} \left|C_{mn}^l\right|^2 [l(l+1)]^r < \infty. \tag{2}$$

Let us show that $D(\Delta_2^{r/2}) \subset W_2^r(B, \sigma)$. From the relations

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$$B_1 T^l_{mn} = \alpha_{n+1} T^l_{m,n+1}; \quad B_2 T^l_{mn} = \alpha_n T^l_{m,n-1}; \quad B_3 T^l_{mn} = n T^l_{mn},$$

where $\alpha_n = \sqrt{(l+n)(l-n+1)}$, it follows that

$$\left\|T^l_{mn}(\varphi_1, \theta, \varphi_2)\right\|_{W_2^r(B,\sigma)} \leq C l^r \left\|T^l_{mn}\right\|_{L_2(\sigma)} = C \frac{l^r}{\sqrt{2l+1}}. \tag{3}$$

The constant C depends only on r. Let $u \in D(\Delta_2^{r/2})$. Denote by u_N the sum of all terms of the series (1) with indices $l \leq N$. Taking (3) into account, we obtain

$$\|u - u_N\|^2_{W_2^r(B,\sigma)} \leq \sum_{l=N}^{\infty} \sum_{m,n=-l}^{l} (2l+1) \left\|C^l_{mn} T^l_{mn}\right\|^2_{W_2^r(B,\sigma)} \leq$$

$$\leq C \sum_{l=N}^{\infty} \sum_{m,n=-l}^{l} l^{2r} \left|C^l_{mn}\right|^2.$$

By virtue of the convergence of the series (2), the right-hand side of the inequality tends to zero as $N \to \infty$. Consequently, $u \in W_2^r(B,\sigma)$, and $D(\Delta_2^{r/2}) \subset W_2^r(B,\sigma)$.

The operator Δ_2 can be obtained by closure from the set of generalized spherical polynomials, on which $\Delta_2 = (B_1 B_2 - B_3 + B_3^2)$. It then follows from Lemma 1 that the domain of definition of Δ_2 contains the whole space $W_2^2(B,\sigma)$: $D(\Delta_2) \supset W_2^2(B,\sigma)$. Combining this conclusion with the preceding one, we obtain: $D(\Delta_2) = W_2^2(B,\sigma)$.

Let us now consider the operator $\Delta_2^{1/2}$ and show that $D(\Delta_2^{1/2}) = W_2^1(B,\sigma)$. It is easy to verify that, on generalized spherical polynomials,

$$\left\|\Delta_2^{1/2} u\right\|^2_{L_2(\sigma)} = \left\|\frac{\partial u}{\partial \theta}\right\|^2_{L_2(\sigma)} + \left\|\frac{1}{\sin\theta}\left(\frac{\partial u}{\partial \varphi_1} - \cos\theta \frac{\partial u}{\partial \varphi_2}\right)\right\|^2_{L_2(\sigma)} + \left\|\frac{\partial u}{\partial \varphi_2}\right\|^2_{L_2(\sigma)}.$$

Taking into account that

$$\frac{\partial u}{\partial \theta} = \frac{1}{2i}\left(e^{i\varphi_2} B_1 + e^{-i\varphi_2} B_2\right) u; \quad \frac{\partial u}{\partial \varphi_2} = -i B_3 u; \quad \frac{1}{\sin\theta}\left(\frac{\partial u}{\partial \varphi_1} - \cos\theta \frac{\partial u}{\partial \varphi_2}\right) = \frac{1}{2}\left(e^{-i\varphi_2} B_2 - e^{i\varphi_2} B_1\right) u,$$

we obtain

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$$\left\|\Delta_2^{1/2} u\right\|_{L_2(\sigma)} \leq C \left\|u\right\|_{W_2^1(B,\sigma)}.$$

Hence it follows that $D(\Delta_2^{1/2}) \supset W_2^1(B,\sigma)$, and therefore $D(\Delta_2^{1/2}) = W_2^1(B,\sigma)$. The arguments which we have carried out for $r = 1$ and $r = 2$ are easily repeated for any integer $r > 2$. This implies the validity of the following theorem:

Theorem 1. The sets $D(\Delta_2^{r/2})$ and $W_2^r(B,\sigma)$ coincide.

3°. The derivatives of the function $u(\varphi_1, \theta, \varphi_2)$ with respect to $\varphi_1, \theta, \varphi_2$ are expressible in terms of B_1, B_2 and B_3 by the formulas

$$\frac{\partial^r u}{\partial \theta^r} = \left(-\frac{i}{2}\right)^r \left(e^{i\varphi_2} B_1 + e^{-i\varphi_2} B_2\right)^r u; \quad \frac{\partial^r u}{\partial \varphi_2^r} = (-i)^r B_3^r u;$$

$$\frac{\partial^r u}{\partial \varphi_1^r} = \frac{1}{2^r} \left(\sin\theta \cdot e^{-i\varphi_2} B_2 - \sin\theta \cdot e^{i\varphi_2} B_1 - 2i\cos\theta \cdot B_3\right)^r u.$$

In computing $\partial^r u/\partial\theta^r$, the factors $e^{i\varphi_2}$ and $e^{-i\varphi_2}$ are carried outside the sign of the operator

$$\frac{\partial}{\partial\theta} = -\frac{i}{2}\left(e^{i\varphi_2} B_1 + e^{-i\varphi_2} B_2\right),$$

therefore $\partial^r u/\partial\theta^r$ will be equal only

linear combination of expressions of the form

$$e^{i\gamma_1\varphi_2} e^{-i\gamma_2\varphi_2} B_1^{k_1} B_2^{k_2} B_3^{k_3} u = e^{i(\gamma_1-\gamma_2)\varphi_2} B_1^{k_1} B_2^{k_2} B_3^{k_3} u,$$

where $\gamma_1 + \gamma_2 = r,\ k_1 + k_2 + k_3 \leq r$. Similarly, $\partial^r u/\partial\varphi_1^r$ is a linear combination of expressions of the form

$$e^{i(\gamma_1-\gamma_2)\varphi_2} \sin^{\gamma_1+\gamma_2}\theta \cos^{\gamma_3}\theta \cdot B_1^{k_1} B_2^{k_2} B_3^{k_3} u \quad (\gamma_1+\gamma_2+\gamma_3 = r;\ k_1+k_2+k_3 \leq r).$$

Hence we obtain estimates of the norms of the derivatives with respect to $\varphi_1, \theta, \varphi_2$:

$$\left\|\frac{\partial^r u}{\partial\varphi_1^{k_1} \partial\varphi_2^{k_2} \partial\theta^{k_3}}\right\|_{L_2(\sigma)} < C\|u\|_{W_2^r(B,\sigma)}.$$

From Theorem 1 and these estimates the following theorem follows:

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Theorem 2. If the function $u(\varphi_1, \theta, \varphi_2) \in W_2^r(B, \sigma)$, then the series obtained from the expansion of the function u in a Fourier series in generalized spherical functions, after r-fold termwise differentiation with respect to $\varphi_1, \theta, \varphi_2$, converges in $L_2(\sigma)$.

By the symbol D^r we shall denote any derivative of order k with respect to $\varphi_1, \theta, \varphi_2$.

Lemma 2. For generalized spherical functions and their derivatives the estimate

$$\left|D^k T_{mn}^l(\varphi_1, \theta, \varphi_2)\right| \leq O\left(l^{1/2+k}\right) \|T_{mn}^l\|_{L_2(\sigma)}$$

is valid.

We shall not give the proof of Lemma 2, in view of its cumbersomeness.

Theorem 3. If the function $u(\varphi_1, \theta, \varphi_2) \in W_2^r(B, \sigma)$ and $r \geq 3$, then its Fourier series in generalized spherical functions and the series obtained from it by termwise differentiation with respect to $\varphi_1, \theta, \varphi_2$ of order $< r-2$ converge absolutely and uniformly.

It follows from Lemma 2 that the series

$$\sum_{l=0}^{\infty} \sqrt{2l+1} \sum_{m,n=-l}^{l} \left|C_{mn}^l D^k T_{mn}^l(\varphi_1, \theta, \varphi_2)\right|$$

is majorized by the series

$$C \sum_{l=0}^{\infty} \sum_{m,n=-l}^{l} \left|l^r C_{mn}^l\right| l^{-(r-k-1/2)} \leq \frac{C}{2} \sum_{l=0}^{\infty} \sum_{m,n=-l}^{l} \left|l^r C_{mn}^l\right|^2 + \frac{C}{2} \sum_{l=0}^{\infty} \beta_l l^{-(2r-2k-1)}, \tag{4}$$

where $C = \text{const}$, $\beta_l = (2l+1)^2 = O(l^2)$ is the number of linearly independent generalized spherical functions of order l. From (2) and Theorem 1 it follows that the first series in (4) converges. For $r \geq 3$ and $k < r-2$ the second series also converges. Theorem 3 is proved.

Remark. Theorems 2 and 3 remain valid if termwise differentiation of the Fourier series is replaced by termwise application of the operators B_1, B_2, and B_3 to the Fourier series.

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Voronezh Technological
Institute

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Note: Figure translations are in progress. See original paper for figures.

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