ESTIMATION OF POLYNOMIALS IN THE COMPLEX PLANE

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Abstract

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ESTIMATION OF POLYNOMIALS IN THE COMPLEX PLANE

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In the work ([1]) of E. V. Voronovskaya, the extremal properties of the Chebyshev and Zolotarev polynomials on the disk of unit radius of the complex plane were studied, and the validity of analogous properties on a disk of arbitrary radius was also noted. In the present article the authors give a complete solution of the problem posed in the note ([1]), namely: among the set of algebraic polynomials with real coefficients under the condition $\max_{[0,1]} |P_n(x)| = 1$ (reduced), to find that one which at the point $z = \rho e^{i\varphi}$ has the greatest, in modulus, either real part or imaginary part.

It suffices to take $0 < \varphi < \pi$.

Theorem 1. *If the interval-functionals on the set $\{P_n(x)\}$ are denoted in the following way:*

$$F_{\cos} = 1,\ \rho\cos\varphi,\ \rho^2\cos 2\varphi, \dots, \rho^n\cos n\varphi = (\mu_k)_0^n;$$

$$F_{\sin} = 0,\ \rho\sin\varphi,\ \rho^2\sin 2\varphi, \dots, \rho^n\sin n\varphi = (\nu_k)_0^n,$$

then the number of true nodes ([1]) of $F_{\cos}$ and $F_{\sin}$ satisfies the condition $s \geq n$ for any $\rho > 0$ and any $0 < \varphi < \pi$.

Indeed, μ_k and ν_k have the form $\Delta_1(e^{\varphi i})^k + \Delta_2(e^{-\varphi i})^k$, where $\Delta_1 = 1/2$ or $1/2i$ and $\Delta_2 = 1/2$ or $-1/2i$, i.e. (μ_k) and (ν_k) have two fictitious nodes ([2]). But then $s + 2 > n + 1$ ([2]), and the theorem is proved.

Corollary. The extremal polynomials for $F_{\cos}$ and $F_{\sin}$ are either $\pm T_n(x)$, where

$$T_n(x) = \cos n \arccos(2x - 1),$$

or the passport polynomials $[n, n, p]$ ([2,3]), where $p = 0$ or 1; moreover, at each point (ρ, φ) the extremal polynomial is unique.

Let us find necessary and sufficient conditions under which $F_{\cos}$ is served by the polynomials $\pm T_n(x)$ with deviation points $(\tau_k)_0^n$. Expanding $F_{\cos}$ in the nodes $(\tau_k)_0^n$, where $\tau_k = \sin^2(k\pi/2n)$, i.e. solving the system of equations

$$\sum_{p=0}^{n} \delta_p \tau_p^k = \mu_k \quad (k = 0, 1, \dots, n),$$

we obtain for the loads (δ_p) the formula

$$\delta_p = (-1)^{n-p} \frac{\frac{1}{2}\left[\prod_{k\neq p}(\rho e^{\varphi i} - \tau_k) + \prod_{k\neq p}(\rho e^{-\varphi i} - \tau_k)\right]}{\prod_{k\neq p}|\tau_p - \tau_k|} \quad (p = 0, 1, \dots, n).$$

Here the numerator is

$$A_p = \mathrm{Re} \prod_{k\neq p}(\rho e^{\varphi i} - \tau_k).$$

The signs of (δ_p) will alternate if either $A_p \geq 0$, or $A_p \leq 0$ for all p.

Put

$$\arg \prod_{k\neq p}(\rho e^{\varphi i} - \tau_k) = \sum_{k\neq p} \arg(\rho e^{\varphi i} - \tau_k) = \sum_{k\neq p} \varphi_k = \psi_p.$$

Thus, the following is valid:

Theorem 2. *If E_T is the set of points in (z) at which $F_{\cos}$ is served by the polynomials $+T_n(x)$ or $-T_n(x)$, then a necessary and sufficient condition for $\rho e^{\varphi i} \in E_T$ is that either all ψ_p lie in $[-\pi/2, +\pi/2]$ (right half-plane), or in $[\pi/2, 3\pi/2]$ (left half-plane); moreover, in the first case $+T_n(x)$ serves, and in the second $-T_n(x)$.*

Corollary. Fixing ρ and putting $\varphi = \varphi_0$, we have $\varphi_0 < \varphi_1 < \dots < \varphi_n$ and $\psi_n < \psi_{n-1} < \dots < \psi_0$. The arguments of the boundaries of the Chebyshev arc (on the circle of radius ρ) are obtained when ψ_0 or ψ_n touches the imaginary axis. Denote these arguments by a and b, respectively; then for $\varphi_0 = a$ the node $\tau_0 = 0$ drops out (i.e. $\delta_0 = 0$), and for $\varphi_0 = b$ the node $\tau_n = 1$ drops out (i.e. $\delta_n = 0$). According to the theorem on continuous deformation ([2]), the family $F_{\cos}$ is served by the polynomials of passport $[n, n, 0]$ and only by them, i.e. by the Zolotarev polynomials ([3]).

For any $\rho \geq 1$, the arguments of the boundaries of the Zolotarev arcs (α_k, β_k) and their number on the semicircle $(0, \pi)$ are determined from the conditions that ψ_0 (or ψ_k) is equal to $\pi/2, 3\pi/2, \dots, (2n-1)\pi/2$; thus, on the semicircle there are altogether n Zolotarev arcs.

Remark 1. For $F_{\sin}$ analogous results are obtained; thus, $F_{\sin}$ is served by the polynomials $\pm T_n(x)$ if and only if all $(\psi_p)_0^n$ lie either in the upper half-plane or in the lower; otherwise $F_{\sin}$ is served by Zolotarev polynomials.

Let us note some elementary formulas for any $z = \rho e^{\varphi i}$ in the upper half-plane. Put $\overline{\varphi}_k = \varphi_k - \varphi_0 = \psi_0 - \psi_k$. Then

$$\overline{\varphi}_k = \frac{\pi}{2} - \frac{\varphi_0}{2} - \operatorname{arctg}\left[\frac{\rho - \tau_k}{\rho + \tau_k}\operatorname{ctg}\frac{\varphi_0}{2}\right], \tag{1}$$

and further

$$\psi_0 = n\varphi_0 + \sum_{k=1}^{n}\overline{\varphi}_k, \tag{2}$$

$$\psi_n = n\varphi_0 + \sum_{1}^{n}\overline{\varphi}_k - \frac{\pi}{2} + \frac{\varphi_0}{2} + \operatorname{arctg}\left[\frac{\rho - 1}{\rho + 1}\operatorname{ctg}\frac{\varphi_0}{2}\right]. \tag{3}$$

It is obvious that

$$\max_{(\varphi_0)}\overline{\varphi}_k = \arcsin\frac{\tau_k}{\rho} \quad (k = \text{const},\ \varphi_k = \pi/2,\ \rho > 1).$$

In $\sum_1^n \overline{\varphi}_k$ the largest term is $\overline{\varphi}_n \leq \arcsin\frac{1}{\rho}$; thus (for $n = \text{const}$) we have

$$\sum_{1}^{n}\overline{\varphi}_k < n\arcsin\frac{1}{\rho}, \qquad \lim_{\rho\to\infty}\sum_{1}^{n}\overline{\varphi}_k = 0. \tag{4}$$

Theorem 3. *On the semicircle $0 < \varphi_0 < \pi$, as ρ increases, the arguments of the boundaries of the k-th Zolotarev arc ($\alpha_k < \beta_k$) tend to $(2k-1)\pi/2n$.*

Indeed, according to the corollary to Theorem 2, for $\varphi_0 = \alpha_k$ we have from (2)

$$n\alpha_k + \sum_{p=1}^{a}\overline{\varphi}_p(\alpha_k) = \frac{2k-1}{2}\pi,$$

and for $\varphi_0 = \beta_k$ from (3)

$$n\beta_k + \sum_{p=1}^{n}\overline{\varphi}_p(\beta_k) - \frac{\pi}{2} + \frac{\beta_k}{2} + \operatorname{arctg}\left[\frac{\rho - 1}{\rho + 1}\operatorname{ctg}\frac{\beta_k}{2}\right] = \frac{(2k-1)\pi}{2}.$$

According to (4), we obtain

$$\lim_{\rho\to\infty}\alpha_k = \lim_{\rho\to\infty}\beta_k = \frac{(2k-1)\pi}{2n}.$$

Theorem 4. *In the plane (z), the branches-boundaries of the Zolotarev regions $\rho e^{\alpha_k(\rho)i}$ have a system of asymptotes issuing from the point $(n+1)/2n$ on the axis*

Ox, and for the branches $\rho e^{\beta_k(\rho)i}$ the asymptotes are rays issuing from the point $(n-1)/2n$ (the angles of the asymptotes are determined in Theorem 3).

Indeed, assuming that $l(\alpha_k)$ (respectively $l(\beta_k)$) is the length of the arc of radius ρ between the points $\rho e^{(2k-1)\pi i/2n}$ and $\rho e^{\alpha_k i}$ (or $\rho e^{\beta_k i}$), we have (on the basis of the formulas in Theorem 3):

$$l(\alpha_k) = \rho\left(\frac{2k-1}{2n}\pi - \alpha_k\right) = \frac{\rho}{n}\sum_{p=1}^{n}\overline{\varphi}_p(\alpha_k),$$

$$l(\beta_k) = \rho\left(\frac{2k-1}{2n}\pi - \beta_k\right) = \frac{\rho}{n}\sum_{p=1}^{n-1}\overline{\varphi}_p(\beta_k).$$

The oblique abscissa of the point $\rho e^{\alpha_k i}$ is

$$x_\alpha = \frac{\rho\sin\left(\frac{2k-1}{2n}\pi - \alpha_k\right)}{\sin\frac{2k-1}{2n}\pi} = \rho\,\frac{\sin\left[\frac{1}{n}\sum_1^n\overline{\varphi}_p(\alpha_k)\right]}{\sin\frac{2k-1}{2n}\pi}.$$

Here it is permissible first to pass to

$$\lim_{\rho\to\infty}\left[\rho\sin\frac{1}{n}\sum_1^n\overline{\varphi}_p(\varphi)\right]$$

for $\varphi = \text{const}$, and then to put $\varphi = \frac{2k-1}{2n}\pi$. Since $\sum_{p=1}^n\overline{\varphi}_p(\varphi)\to 0$, we have

$$\lim\frac{\rho}{n}\sum_1^n\overline{\varphi}_p(\varphi) = \lim\rho\sum_1^n\left(\frac{\pi}{2} - \frac{\varphi}{2} - \operatorname{arc\,tg}\frac{\rho-\tau_k}{\rho+\tau_k}\operatorname{ctg}\frac{\varphi}{2}\right) =$$

$$= \sum_{k=1}^n\tau_k\frac{2\operatorname{ctg}(\varphi/2)}{1+\operatorname{ctg}^2(\varphi/2)} = \sin\varphi\sum_1^n\tau_k.$$

Hence $\lim x_\alpha = (n+1)/2n$. In exactly the same way we obtain $\lim x_\beta = (n-1)/2n$, and the theorem is proved.

Corollary. The sum of the Zolotarev arcs on the semicircle of radius ρ tends to $1/n\sin(\pi/2n)$ as $\rho\to\infty$.

Indeed, the length of the k-th Zolotarev arc is

$$l(\alpha_k) - l(\beta_k) = \frac{\rho}{n}\left[\sum_{p=1}^{n}\overline{\varphi}_p(\alpha_k) - \sum_{p=1}^{n-1}\overline{\varphi}_p(\beta_k)\right] \to \frac{1}{n}\sin\frac{(2k-1)\pi}{2n}.$$

For the sum in the limit we have

$$\frac{1}{n}\sum_{k=1}^{n}\sin\frac{2k-1}{2n}\pi = \frac{1}{n\sin(\pi/2n)}.$$

Thus, for large n this sum is close to $2/\pi$.

Remark 2. For $F_{\sin}$, the same method gives analogous results: 1) as $\rho \to \infty$, the arguments of the boundaries of the Zolotarev arc (α_k, β_k) tend to $k\pi/n$ $(k = 1, 2, \dots, n-1)$; the n-th Zolotarev arc, for every ρ, degenerates into the point $\alpha_n = \beta_n = \pi$; 2) as $\rho \to \infty$, the branches $\rho e^{\alpha_k(\rho)i}$ and $\rho e^{\beta_k(\rho)i}$ have asymptotes issuing respectively from the points $(n+1)/2n$ and $(n-1)/2n$ on the axis Ox; 3) the sum of the lengths of the Zolotarev arcs tends to $\mathrm{ctg}(\pi/2n)/n$ as $\rho \to \infty$.

Remark 3. From the properties proved it follows that, for one and the same $\rho \geq 1$, the Zolotarev arcs for $F_{\cos}$ and for $F_{\sin}$ cannot overlap.

In order to consider the boundaries of the Zolotarev regions for $\rho < 1$, let us note, restricting ourselves to the case $F_{\cos}$, that in general the arguments of these boundaries $\alpha(\rho)$

and $\beta(\rho)$ are respectively the roots of the equations

$$F_{\cos}\left[\frac{R_{n+1}(x)}{x}\right] = 0, \qquad F_{\cos}\left[\frac{R_{n+1}(x)}{x-1}\right] = 0, \quad \text{where} \quad R_{n+1}(x) = \prod_{0}^{n}(x-\tau_k).$$

Putting

$$\frac{R_{n+1}(x)}{x} = \sum_{0}^{n} S_{n-k}^{(1)} x^k, \qquad \frac{R_{n+1}(x)}{x-1} = \sum_{0}^{n} S_{n-k}^{(2)} x^k,$$

we have (for any $\rho > 0$)

$$\rho^n \cos n\alpha + S_1^{(1)}\rho^{n-1}\cos(n-1)\alpha + \dots + S_n^{(1)} = 0,$$

$$\rho^n \cos n\beta + S_1^{(2)}\rho^{n-1}\cos(n-1)\beta + \dots + S_{n-1}^{(2)}\rho\cos\beta = 0.$$

Putting: 1) $\alpha = 0$ and 2) $\beta = 0$, we have

$$\prod_{1}^{n}(\rho - \tau_k) = 0$$

and

$$\prod_{0}^{n-1}(\rho - \tau_k) = 0.$$

Thus, the curves $\rho e^{\alpha(\rho)i}$ and $\rho e^{\beta(\rho)i}$ intersect Ox at the points $(\tau_p)_1^n$ and $(\tau_p)_0^{n-1}$, and the Chebyshev domains have on $[0,1]$ "sources" at the points $(\tau_k)_0^n$, while the Zolotarev domains contain the entire segment $[0,1]$, except for (τ_k). The sources of the branches for α_k and β_k are τ_{n-k+1} and τ_{n-k}.

For lack of space we shall confine ourselves to the indicated properties for $\rho < 1$. We are also unable to provide drawings illustrating the distribution of the mentioned domains in the (z)-plane.

Finally, let us note the most immediate applications to classical extremal problems. Denote by $E_{\cos}$ and $E_{\sin}$, respectively, the Chebyshev sets of points in (z) for $F_{\cos}$ and $F_{\sin}$, and by $\mathcal{E}_{\cos}$ and $\mathcal{E}_{\sin}$ the Zolotarev sets.

I. Among all algebraic polynomials $\{P_n(x)\}$ reduced on $[0,1]$ and with real coefficients, find the one which at the point z_0 attains the maximum modulus. If $z_0 \in (E_{\cos}, E_{\sin})$, then this polynomial is, evidently, $\pm T_n(x)$, and $|T_n(z_0)| = N$, the norm of the functional $(z_0)^n$.

Problem I can be rephrased as follows:

I_A. Among all $\{P_n(x)\}$ taking at the point z_0 the value $|P_n(z_0)| = A$ in modulus, find the one which deviates least from zero on $[0,1]$. If the solution of problem I_A is $Y_n(x)$ with deviation L, and $P_n(x)$ is the solution of problem I, then

$$Y_n(x)/L = P_n(x), \qquad A = LN.$$

II. Among all $\{P_n(x)\}$ reduced on $[0,1]$, find the one which at the point z_0 gives: a) either $\max|\operatorname{Re} P_n(z)|$, b) or $\max|\operatorname{Im} P_n(z)|$. If $z_0 \in (E_{\cos}, E_{\sin})$, then the desired polynomial is $\pm T_n(x)$ for both variants a) and b). If $z_0 \in \mathcal{E}_{\cos}$, then the polynomial for variant a) belongs to the family $Z_n(x,\vartheta)$, where ϑ is the leading coefficient; for $z_0 = \rho e^{\varphi i}$ the value of ϑ is found from the condition

$$F_{\cos}[Z_n(x,\vartheta)] = \max(\vartheta),$$

i.e.

$$\partial F_{\cos}(Z_n)/\partial\vartheta = 0$$

([2,4]), or, what is the same,

$$F_{\cos}[R_n(z)] = 0, \qquad \text{where} \qquad R_n(x) = \prod_1^n (x - \sigma_k);$$

$\sigma_k(\vartheta)$ are the nodes of the Zolotarev polynomial; in the case under consideration the extremal polynomial for variant b) is $\pm T_n(x)$ (see Remark 3). For $z_0 \in \mathcal{E}_{\sin}$ analogous solutions are obtained.

Problem II can also be formulated differently: among all $\{P_n(x)\}$ for which $|\operatorname{Re} P_n(z_0)| = A$ or $|\operatorname{Im} P_n(z_0)| = B$, find the one which deviates least from zero on $[0, 1]$.

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CITED LITERATURE

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[4] E. V. Voronovskaya, Proceedings of the Third All-Union Mathematical Congress, **3**, Publishing House of the Academy of Sciences of the USSR, 1958, p. 177.

Note: Figure translations are in progress. See original paper for figures.