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Abstract

Full Text

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On Perfect Bicompact Extensions

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This article is directly connected with the work ([9]), whose terminology, notation, and results are used extensively below.*

1. In a paper by R. Duda ([3]), and then in a somewhat more general form in ([9]), a theorem was proved stating that every homeomorphism of a completely regular space X onto a space Y can be extended to bicompact extensions of these spaces which are perfect and have pointiform remainders. The question arises (it was brought to my attention by Yu. M. Smirnov) whether in this theorem homeomorphisms can be replaced by continuous mappings, in some sense close to homeomorphisms (of course, an arbitrary continuous mapping can be extended only to the Čech extension). We note that, when homeomorphisms are extended, the remainder is mapped into the remainder. Perfect mappings possess the same property.**

Lemma 1. Let f be a continuous mapping of the space X into Y, and let $\bar{f}$ be an extension of f to some bicompact extensions $\bar{X}$ and $\bar{Y}$ of these spaces. Then $\bar{f}(\bar{X} \setminus X) \subset \bar{Y} \setminus Y$ if and only if the mapping f is perfect***.

Proof. If $\bar{f}(\bar{X} \setminus X) \subset \bar{Y} \setminus Y$, then it is obvious that f is perfect. Conversely, let f be perfect, but $\bar{f}x = y \in Y$ for some $x \in \bar{X} \setminus X$. Since $f^{-1}y$ is a bicompact subspace of X, there is in $\bar{X}$ a closed neighborhood V of the point x such that $V \cap f^{-1}y = \emptyset$. Let $U = V \cap X$; then U is a closed set in X, and fU is a closed set in Y containing y, contrary to the fact that $f^{-1}y \cap U = \emptyset$.

The theorem on extending homeomorphisms is generalized as follows:

Theorem 1. Let $\bar{X}$ and $\bar{Y}$ be bicompact extensions of the spaces X and Y, with $\bar{X}$ perfect and $\bar{Y}$ having a pointiform remainder. Every perfect mapping $f : X \to Y$ extends to a mapping
$\bar{f} : \bar{X} \to \bar{Y}$.

Proof. Consider the mappings

$$\bar{X} \xleftarrow{\pi_X} \beta X \xrightarrow{\tilde{f}} \beta Y \xrightarrow{\pi_Y} \bar{Y},$$

where $\tilde{f}$ is the extension of f to the Čech extensions, and π_X and π_Y are the natural projections. Put $\bar{f} : \pi_Y \tilde{f} \pi_X^{-1}$. We shall show that the mapping $\bar{f}$ is single-valued. This is obvious if $x \in X$. If $x \in \bar{X} \setminus X$, then by Theorem 2 of ([9]) the set $\pi_X^{-1}x$ is connected; by Lemma 1, $\tilde{f}\pi_X^{-1}x \subset \beta Y \setminus Y$; since the set $\tilde{f}\pi_X^{-1}x$ is connected and bicompact, and the remainder in $\bar{Y}$ is pointiform, $\pi_Y\tilde{f}\pi_X^{-1}x$ consists of a single point. The mapping $\bar{f}$ is continuous, since it is a superposition of continuous (multivalued) mappings (see ([6])). Finally, it is obvious that $\bar{f}$ is an extension of the mapping f.

Corollary. Let X and Y be completely regular spaces possessing minimal perfect extensions μX and μY. Then every perfect mapping $f : X \to Y$ extends to a mapping
$\bar{f} : \mu X \to \mu Y$.

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* A detailed exposition of the results of ([9]) is contained in ([10]).

** A continuous mapping is called perfect if it is closed and the full preimages of all points are bicompact.

*** This lemma is essentially proved in ([7]), Lemma 4 and Theorem 3, and in ([12]), Lemma 1.5.

The following theorem is a partial converse of Theorem 1.

Theorem 2. *Let $\bar{Y}$ be a bicompact extension of a space Y such that, for every perfect mapping $f : X \to Y$ and every perfect extension $\bar{X}$ of the space X, there is an extension $\bar{f} : \bar{X} \to \bar{Y}$. Then Y has a bicompact extension with a punctiform remainder, preceding the extension $\bar{Y}$.*

Consider as f the identity mapping of the space Y onto itself. Applied to this case, the condition of the theorem gives that the extension $\bar{Y}$ precedes every perfect extension of the space Y. Therefore it suffices to prove the following assertion, which may be regarded as a strengthening of Theorem 3 from ([9]).

Theorem 2′. *If a space Y has a bicompact extension $\bar{Y}$ preceding all its perfect extensions, then it has a minimal perfect extension μY.*

Indeed, let π be the natural projection of βY onto $\bar{Y}$. Since $\bar{Y}$ precedes every perfect extension of the space Y, by Theorem 2 from ([9]) π maps every connected bicompact subset of $\beta Y \setminus Y$ to a single point. Therefore, for every point $y \in \bar{Y} \setminus Y$, the connected components of the set $\pi^{-1}y$ are maximal connected bicompact subsets in $\beta Y \setminus Y$. The subsequent arguments are carried out in the same way as in the proof of Theorem 3 in ([10]).

2. In ([9]) an example was given of a peripherally bicompact space all of whose bicompact extensions with zero-dimensional remainder have dimension greater than the space itself. Independent examples of the same type were proposed by Lelek ([4]) and Nishiura ([5]). In doing so, Lelek uses the following theorem proved by him:

Let X be a dense subset in the sphere S^n, the complement of which separates

S^n at no point, and let Y be a compact extension of the space X with zero-dimensional remainder. Then $\dim Y \geq n$.

Lelek makes the conjecture that this theorem remains valid if, in its formulation, the zero-dimensional remainder is replaced by a punctiform one. The following theorem shows, in particular, that this conjecture is correct.

Theorem 3. *Let Y be a perfect bicompact extension of a space X, $\dim Y = n$, and $\dim \Phi \leq n-1$ for every bicompact subset $\Phi \subset Y \setminus X$. Then, for every bicompact extension Z of the space X having a punctiform remainder, $\dim Z \geq n$.*

We give a proof of this theorem using cohomology theory. Since $\dim Y = n$, there is in Y a closed subset A such that $H^n(Y, A; Z) \neq 0$ (Z is the group of integers). Let $U = Y \setminus A$, and let Z_U be the sheaf over Y equal to zero on A and inducing the constant sheaf Z over U. For every $q \geq 0$ we have

$$H^q(Y, A; Z) = H^q(Y; Z_U).$$

Consequently,

$$H^n(Y; Z_U) \neq 0.$$

By Theorem 3 from ([9]), the extension Z precedes Y. Let f be the natural projection of Y onto Z. Consider the Leray spectral sequence of the mapping f. The limit term E_∞ of this spectral sequence is associated with $H^*(Y; Z_U)$, and its second term has the form

$$E_2^{p,q} = H^p(Z; R^q f Z_U),$$

where $R^q f Z_U$ is the q-th direct image of the sheaf Z_U (see, for example, ([2])). Since for every point $z \in Z$ we have $\dim f^{-1}z \leq n-1$, it follows that

$$E_2^{p,q} = 0$$

for $q \geq n$. Moreover, if $z \in X$, then

$$(R^q f Z_U)_z = 0$$

for $q \geq 1$, i.e. the sheaves $R^q f Z_U$ for $q \geq 1$ are concentrated on the punctiform set $Z \setminus X$.

Lemma 2. *If a sheaf F over a bicompactum B is concentrated on a punctiform set N, then $H^q(B; F) = 0$ for $q \geq 1$.*

Proof. Let c be a q-dimensional cocycle defined on a covering $\{V_i\}$. Since every section of the sheaf that is nonzero on a closed—

zero subset, then the sections defining the cocycle c are equal to zero outside some zero-dimensional set $\Gamma \subset N$. One can find a covering $\{W_j\}$, inscribed in $\{V_i\}$, such that $W_j \cap W_k \cap \Gamma = \emptyset$ for $j \neq k$. Therefore c induces a zero cocycle on the covering $\{W_j\}$.

Thus, returning to the proof of Theorem 3, we have that $E_2^{p,q} = 0$ if $pq > 0$, and $E_2^{0,q} = 0$ for $q \geq n$. Therefore $E_2^{n,0} \neq 0$, since otherwise we would have

$$\sum_{p+q=n} E_\infty^{p,q} = 0,$$

contrary to the fact that $H^n(Y; Z_U) \neq 0$. Thus $H^n(Z; R^0 f Z_U) \neq 0$, and hence $\dim Z \geq n$.

3. In ([11]), as a generalization of Freudenthal' s π-bicompact bases (see ([9])), the notion of a normal base was introduced. An open base $\mathcal{B} = \{U\}$ of a space X is called normal if: 1) from $U \in \mathcal{B}$ it follows that $X \setminus [U] \in \mathcal{B}$; 2) from $U_1, U_2 \in \mathcal{B}$ it follows that $U_1 \cap U_2 \in \mathcal{B}$; 3) if O is an open set in X, $U_1 \in \mathcal{B}$, and $[U_1] \subset O$, then there is $U_2 \in \mathcal{B}$ such that $[U_1] \subset U_2 \subset [U_2] \subset O$. Every normal base defines a bicompact extension of the space X, corresponding to the proximity relation defined as follows: closed sets A and B are considered far apart if there exists $U \in \mathcal{B}$ such that $A \subset U$ and $[U] \cap B = \emptyset$. Further study of normal bases was undertaken by Banaschewski and Maranda ([1]), who proposed the problem of describing the extensions corresponding to normal bases. The following theorem answers this question for the case of countable normal bases.

Theorem 4. *If a space has a countable normal base, then it is peripherally bicompact, and the extension corresponding to such a base is an extension with zero-dimensional remainder corresponding to some π-bicompact base.*

Thus, a space with the second axiom of countability is peripherally bicompact if and only if it has a countable normal base.*

Proof. Let $\mathcal{B}'$ be the collection of those sets of the base $\mathcal{B}$ which are canonical. Then $\mathcal{B}'$ is also a normal base, which, obviously, defines the same proximity as $\mathcal{B}$. Let $\overline{X}$ be the corresponding bicompact extension. We shall show that $\mathcal{B}'$ is a π-bicompact base. Let $U \in \mathcal{B}'$, $F = \mathrm{Fr}_X U$, and $x \in \overline{X} \setminus [F]$. Since U is a canonical set, $x \in \overline{X} \setminus [X \setminus [U]]$. Therefore in $X \setminus [U]$ there is a sequence $\{x_n\}$ converging to x. If $x \notin F$, then the set $\{x_n\}$ is closed in X and does not intersect $[U]$. But then $\{x_n\}$ is far from $[U]$, contrary to the fact that in $\overline{X}$ they have the common limit point x. Thus, $\overline{X}[F] = F$, i.e. F is a compact set.

The following lemma, proved in ([1]), reveals an important property of normal bases.

Lemma 3. *The bicompact extension corresponding to a normal base is perfect with respect to all sets of this base.*

Proof. We apply Lemma 1 from ([9]). Let $U \in \mathcal{B}$ and $A \subset U$, with $A\,\delta\,\mathrm{Fr}_X U$. This means that there exists $V \in \mathcal{B}$ such that $[A] \subset V$ and $[V] \cap \mathrm{Fr}\,U = \emptyset$. But then $W = V \cap U \in \mathcal{B}$, with $[A] \subset W$ and $[W] \subset U$. This means that $A\,\bar{\delta}\,(X \setminus U)$, as was required to prove.

Lemma 3 leads to the following definition. A base $\mathcal{B} = \{U\}$ of a space X will be called a perfect base of a bicompact extension $\overline{X}$ if: 1) the sets of the form

$O\langle U\rangle$, $U \in \mathcal{B}$, form a base in $\overline{X}$; 2) the extension $\overline{X}$ is perfect with respect to all sets $U \in \mathcal{B}$.

By virtue of Lemma 3, every normal base is a perfect base of the bicompact extension it defines. However, not every bicompact extension has a perfect base. For example, let X be the di-

* Of course, this theorem does not extend to normal bases of arbitrary cardinality, since in every normal space the base consisting of all open sets is normal.

is a discrete set, and $\bar{X}$ is its bicompact extension, having positive dimension. Then $\bar{X}$, obviously, has no perfect base. Nevertheless, as the following theorem shows, in contrast to normal bases, extensions with a perfect base occur quite often.

Theorem 5. *Let Y be an arbitrary bicompact extension of the space X. There exists a bicompact extension Z, lying after Y, having the same weight as Y, and possessing a perfect base.*

Proof. Let τ be the weight of the bicompactum Y. Consider on the space X a uniform structure Σ, consisting of finite open coverings, which corresponds to the bicompact extension Y, has cardinality τ, and whose coverings are ordered in such a way that for any coverings $\alpha, \beta \in \Sigma$, from $\alpha < \beta$ it follows that $\alpha <{}^*\beta^{\frown *}$, with respect to this ordering Σ is a directed set, and for every $\beta \in \Sigma$ there are in Σ only finitely many coverings $\alpha < \beta$. The existence of such a structure is proved in ([8]), Lemma 1. The number of coverings in Σ preceding the covering α will, as in ([8]), be called the number of the covering α.

Construct a sequence of structures $\Sigma_0, \Sigma_1, \ldots$, the coverings of which will be in one-to-one correspondence with the coverings of Σ. This correspondence will induce on the structures Σ_i an order relation satisfying the requirements listed above. Put $\Sigma_0 = \Sigma$. Suppose that the structure Σ_{k-1} has already been constructed. Linearly order the set of elements of all coverings from Σ_{k-1} having numbers $\leq k-1$. Let α be any covering from Σ_{k-1} with number $\geq k$, and let $\{V_1, \ldots, V_N\}$ be the ordered aggregate of all elements of coverings with number $\leq k-1$ preceding α. We refine every element $U \in \alpha$ in the following way. If $U \cap \operatorname{Fr} V_1 \neq \emptyset$, then we leave U unchanged; but if $U \cap \operatorname{Fr} V_1 = \emptyset$, then we replace U by the sets $U \cap V_1$, $U \cap (X \setminus [V_1])$. Then we do the same with the obtained sets and the element V_2, and so on, until we reach V_N. Then with the obtained sets we return to V_1, V_2, and so on. Since as a result of all these transformations the set U can be split into no more than 2^N parts, at some step this process stabilizes. Replacing in this way all coverings from Σ_{k-1} having number $\geq k$ by the coverings thus constructed, we obtain a system of coverings Σ_k, which, as is not difficult to verify, is a structure. Since, in passing from Σ_{k-1} to Σ_k, the coverings with numbers $\leq k-1$ do not change, the "limit" structure Σ_∞ is naturally defined. The structure Σ_∞ is finer than Σ and has the following property: if $\alpha, \beta \in \Sigma_\infty$, $\alpha < \beta$, then for any sets $U \in \alpha$, $V \in \beta$ either $V \cap \operatorname{Fr} U \neq \emptyset$, or one of the inclusions $V \subset U$, $V \subset X \setminus [U]$ holds. From the latter property (with the aid of Lemma 1 from ([9])) it follows that the bicompact extension Z corresponding to the structure Σ_∞ is perfect with respect to all elements of the coverings of this structure. Thus, we have obtained a bicompact extension Z, lying after the extension Y, having the same weight and possessing a perfect base.

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* The notation $\alpha <^* \beta$ means that the covering β is star-inscribed in α.

Note: Figure translations are in progress. See original paper for figures.