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# MATHEMATICS

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## Abstract

## Full Text

MATHEMATICS

V. N. LATYSHEV

# ON ALGEBRAS WITH IDENTICAL RELATIONS

(Presented by Academician P. S. Aleksandrov, 5 V 1962)

In § 1 of the present note we shall assume known the concepts and results of the work of Specht <sup>(1)</sup>. Let  $T$  be a  $T$ -ideal of the free algebra with identity over a field  $K$  of characteristic zero, and let  $\Gamma_n$  be the module of all proper  $n$ -linear forms over  $K$ . Denote  $T_n = \Gamma_n \cap T$ . The modules  $T_n$  completely determine the  $T$ -ideal  $T$  <sup>(1)</sup>. In each module  $T_n$ , as in a linear space over  $K$ , one can choose a basis; the union of the bases of the modules  $T_n$  will be called a basis of the  $T$ -ideal  $T$ . Denote by  $T^2$  the  $T$ -ideal generated by the commutator  $[u, v] = uv - vu$ ; then  $T_n^2 = \Gamma_n \cap T^2 = \Gamma_n$ . It is clear that the union of the Specht bases <sup>(1)</sup> is a basis of the  $T$ -ideal  $T^2$ . Denote by  $T^3$  the  $T$ -ideal generated by the commutator of the third degree  $[u, v, w] = [[u, v], w]$ . In § 1 a certain basis of the  $T$ -ideal  $T^3$  is indicated. The result obtained makes it possible, "from general considerations," to give a negative solution of Kaplansky's hypothesis <sup>(2)</sup> on the possibility of embedding a  $PI$ -algebra in a full matrix algebra over a commutative algebra. Earlier this problem was solved by Cohn in an unpublished work (see <sup>(3)</sup>). In § 2 special Lie algebras with identical relations are studied ( $SPI$ -Lie algebras), i.e. Lie algebras admitting an exact embedding in an associative  $PI$ -algebra. A positive solution is given in the class of  $SPI$ -Lie algebras to a problem analogous to A. G. Kurosh's problem in associative algebras.

1. Introduce the notation:  $G_n$  is the submodule in  $\Gamma_n$  generated by products of commutators in the variables  $x_i$ , in which, as a factor, at least once there necessarily occurs a commutator in  $x_i$  of degree higher than two,  $u_1 = [x_1, x_2][x_3, x_4] \cdots [x_{2k-1}, x_{2k}]$ . It is clear that  $G_n \subset T^3$ . Let

$$w = [x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}] \cdots [x_{i_{2k-1}}, x_{i_{2k}}] \in \Gamma_{2k}.$$

We shall call the composition of the element  $w$  the set of unordered pairs of indices  $(i_l, i_{l+1})$  such that the commutator  $[x_{i_l}, x_{i_{l+1}}]$  occurs in the writing of  $w$ . There is one and only one element of the Specht basis <sup>(1)</sup> among the elements of the form  $w$  of a given composition. Number all compositions in an arbitrary order, taking the composition of the element  $u_1$  as the first. From the set of elements of the form  $w$  of composition with number  $t$ , choose the elements of the Specht basis

$$u_t = [x_{\xi_1}, x_{\xi_2}][x_{\xi_3}, x_{\xi_4}] \cdots [x_{\xi_{2k-1}}, x_{\xi_{2k}}]$$

and form the element  $z_t = u_1 - (-1)^{\varepsilon_t} u_t$ ,  $t > 1$ , where  $\varepsilon_t$  is the number of inversions in the permutation  $\xi_1 \dots \xi_{2k}$  of the numbers  $1, \dots, 2k$ . Since the  $u_t$  are elements of the Specht basis, the  $z_t$  are linearly independent.

**Theorem 1.** The elements of the Specht basis in  $\Gamma_{2k+1}$  form a basis of the module  $T_{2k+1}^3$ . In the module  $T_{2k}^3$ , a basis is formed by the elements  $z_t$  and the elements of the Specht basis contained in  $G_{2k}$ .

**Proof.** The first assertion is trivial, since

$$T_{2k+1}^3 = \Gamma_{2k+1} \cap T^3 = \Gamma_{2k+1} = G_{2k+1} \subset T^3.$$

For the proof of the second assertion we introduce notation:  $F_n$  is the module of all  $n$ -linear forms,  $n = 2k$ ;  $Q_n$  is the submodule in  $F_n$  generated by elements of the form  $p\tau q$ , where  $p$  and  $q$  are words in  $x_i$  and 1, and  $\tau$  is an element of one of the following forms:

$$\begin{aligned} & [x_i, x_j][x_k, x_l] + [x_i, x_k][x_j, x_l], \\ & [x_i, x_j][x_k, x_l] + [x_l, x_j][x_k, x_i], \\ & [x_i, x_j][x_k, x_l] + [x_k, x_j][x_i, x_l], \\ & [x_i, x_j][x_k, x_l] + [x_i, x_l][x_k, x_j]. \end{aligned} \tag{1}$$

Put  $M_n = \Gamma_n \cap Q_n$ ,  $\mathfrak{M}_n = M_n + G_n$ .

**Lemma 1.** The equality  $\mathfrak{M}_n = T_n^3$  holds.

**Proof.** We have:  $[uv, r, s] = [[u, r]v + u[v, r], s] = [u, r, s]v + u[v, r, s] + [u, r][v, s] + [u, s][v, r]$ , whence  $[u, r][v, s] + [u, s][v, r] \equiv 0 \pmod{T^3}$ , and elements of the form (1) lie in  $T^3$ ; but this means that  $M_n \subset \Gamma_n \cap T^3 = T_n^3$ , and since  $G_n \subset T_n^3$ , we have  $\mathfrak{M}_n = M_n + G_n \subset T_n^3$ .

Let us prove the reverse inclusion. Let  $a \in F_n \cap T^3$ ; then there is a representation of the form  $a = \sum \alpha p[u, v, r]q$ , where  $\alpha \in K$ ;  $p, q, u, v, r$  are words in the  $x_i$  and 1. For definiteness put  $r = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_k}$ ,  $v = x_{\beta_1} x_{\beta_2} \cdots x_{\beta_l}$ ,  $u = x_{\gamma_1} x_{\gamma_2} \cdots x_{\gamma_m}$ . We shall have

$$\begin{aligned} p[u, v, r]q &= \sum_{\nu=1}^k p x_{\alpha_1} \cdots x_{\alpha_{\nu-1}} [u, v, x_{\alpha_\nu}] x_{\alpha_{\nu+1}} \cdots x_{\alpha_k} q = \\ &= \sum_{\nu=1}^k p x_{\alpha_1} \cdots x_{\alpha_{\nu-1}} [u, x_{\alpha_\nu}, v] x_{\alpha_{\nu+1}} \cdots x_{\alpha_k} q = \end{aligned}$$

$$\begin{aligned}
 & - \sum_{\nu=1}^k p x_{\alpha_1} \cdots x_{\alpha_{\nu-1}} [v, x_{\alpha_\nu}, u] x_{\alpha_{\nu+1}} \cdots x_{\alpha_k} q = \\
 & = \sum_{\nu=1}^k \sum_{\mu=1}^l p x_{\alpha_1} \cdots x_{\alpha_{\nu-1}} x_{\beta_1} \cdots x_{\beta_{\mu-1}} [u, x_{\alpha_\nu}, x_{\beta_\mu}] x_{\beta_{\mu+1}} \cdots x_{\beta_l} x_{\alpha_{\nu+1}} \cdots x_{\alpha_k} q - \\
 & - \sum_{\nu=1}^k \sum_{\pi=1}^m p x_{\alpha_1} \cdots x_{\alpha_{\nu-1}} x_{\gamma_1} \cdots x_{\gamma_{\pi-1}} [v, x_{\alpha_\nu}, x_{\gamma_\pi}] x_{\gamma_{\pi+1}} \cdots x_{\gamma_m} x_{\alpha_{\nu+1}} \cdots x_{\alpha_k} q.
 \end{aligned}$$

Denote by  $G_n^*$  the submodule in  $F_n$  generated by elements of the form  $p[x_i, x_j, x_k]q$ ; clearly,  $G_n^* \cap \Gamma_n = G_n$ . Since the commutator of generators in the elements  $F_n$  is commutable with any generator modulo  $G_n^*$ , we easily obtain

$$p[v, x_i, x_j]q = \sum ([x_{\alpha_s}, x_i][x_{\alpha_t}, x_j] + [x_{\alpha_s}, x_j][x_{\alpha_t}, x_i])R \pmod{G_n^*}. \quad (3)$$

Let  $a \in \Gamma_n \cap T^3 = T_n^3$ ; from (2) and (3) it follows that  $a \in Q_n + G_n^*$ , i.e.  $a = b + c$ ,  $b \in Q_n$ ,  $c \in G_n^*$ . To the right-hand side of the equality  $a = b + c$  we apply the Specht algorithm: “moving” the variables  $x_1, x_2, \dots$  to the right with successive formation of commutators (1); we shall have:  $a = b^* + c^*$ ,  $b^* \in M_n$ ,  $c^* \in G_n$ , i.e.  $a \in M_n + G_n = \mathfrak{M}_n$ , and  $T_n^3 \subset \mathfrak{M}_n$ . Lemma 1 is proved.

**Lemma 2.** The elements  $z_t \in M_n$ . Moreover, the residue classes of the elements  $z_t$ ,  $t > 1$ , form a basis of the quotient module  $M_n/M_n \cap G_n$ .

**Proof.** On the set of linear combinations of elements of the form  $w = [x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}] \cdots [x_{i_{n-1}}, x_{i_n}]$  define linear operators  $\sigma_{ij}$ : the element  $w\sigma_{ij}$  is obtained from  $w$  by changing the signs and transposing the variables  $x_i$  and  $x_j$ . It can be shown that  $w - w\sigma_{ij} \in M_n$ , whence it follows that if  $\Sigma = \sigma_{i_1 j_1} \sigma_{i_2 j_2} \cdots \sigma_{i_m j_m}$ , then  $w - w\Sigma \in M_n$ . For any  $t > 1$  one can indicate such a product  $\Sigma$  of the operators  $\sigma_{ij}$  that  $u_1 \Sigma = (-1)^{\varepsilon_t} u_t$  and  $z_t = u_1 - (-1)^{\varepsilon_t} u_t = u_1 - u_1 \Sigma \in M_n$ . Now let us show that every element of  $M_n$  is linearly expressible modulo  $G_n$  in terms of the elements  $z_t$ ,  $t > 1$ . We shall call the commutator  $[x_i, x_j]$  improper if  $i > j$ . Denote by  $\varepsilon(w)$  the number of improper commutators occurring in the notation of  $w$ ; then  $w \equiv (-1)^{\varepsilon(w)} u_t \pmod{G_n}$ , where  $u_t$  is the element of the Specht basis with the same composition as  $w$ . Denote by  $\pi(w)$  the permutation of the indices  $i_1, i_2, \dots, i_n$ ;  $\pi(u_t)$  has the analogous meaning. The parities of  $\pi(w)$  and  $\pi(u_t)$  coincide if and only if  $\varepsilon(w)$  is even. From the form of the elements (1) it easily follows that the elements of the module  $M_n$  are linearly expressed through elements of the form  $\theta = w - w\sigma_{ij} = w + w'$ , where  $w' = -w\sigma_{ij}$ . Let the composition number of  $w$  be  $t$  and

the number of the component  $w'$  is  $s$ . We shall have:

$$\begin{aligned} u_t &\equiv (-1)^{\varepsilon(w)} w \pmod{G_n}, & w' &\equiv (-1)^{\varepsilon(w')} u_s \pmod{G_n}, \\ z_t &= u_1 - (-1)^{\varepsilon_t} u_t \equiv u_1 - (-1)^{\varepsilon(w)+\varepsilon_t} w \pmod{G_n}, \\ z_s &= u_1 - (-1)^{\varepsilon_s} u_s \equiv u_1 - (-1)^{\varepsilon(w')+\varepsilon_s} w' \pmod{G_n}, \end{aligned}$$

whence

$$z_s - z_t \equiv (-1)^{\varepsilon(w)+\varepsilon_t} w - (-1)^{\varepsilon(w')+\varepsilon_s} w' \pmod{G_n}.$$

Obviously,  $\pi(w)$  and  $\pi(w')$  have different parities; therefore the numbers  $\varepsilon(w) + \varepsilon_t$  and  $\varepsilon(w') + \varepsilon_s$  also have different parities, and  $\theta \equiv \pm(z_s - z_t) \pmod{G_n}$ , while the elements  $z_t$ ,  $t > 1$ , form a basis of  $M_n$  modulo  $G_n$ . Lemma 2 is proved.

From Lemmas 1 and 2 Theorem 1 follows trivially, as was required to prove.

**Corollary 1.** A  $T$ -ideal containing the commutator  $[u, v, w]$  has a finite number of generators.

The assertion follows easily from a comparison of the dimensions of  $\Gamma_n$  and  $T_n^3$ .

**Corollary 2.** The universal algebra  $A$  corresponding to the identity  $[x_1, x_2, x_3] = 0$  cannot be embedded in a full matrix algebra over a commutative algebra.

Indeed, for the standard identity one can establish the relation:

$$\begin{aligned} S_{n+2}(x_1, \dots, x_{n+2}) &= \\ &= \sum_{1 \leq i < k \leq n+2} (-1)^{k+i-1} [x_i, x_k] S_n(x_1, \dots, \hat{x}_i, \dots, \hat{x}_k, \dots, x_{n+2}), \end{aligned}$$

where the sign  $\wedge$  denotes omission of the corresponding argument. Hence it is not difficult to obtain the congruence:

$$S_{2k}(x_1, \dots, x_{2k}) \equiv C_2^2 C_4^2 C_6^2 \dots C_{2k}^2 [x_1, x_2][x_3, x_4] \dots [x_{2k-1}, x_{2k}] \pmod{T^3};$$

but this means that no standard identity is satisfied in  $A$ .

2. A Lie algebra  $L$  over a field  $K$  is called a special  $PI$ -algebra of Lie (an  $SPI$ -algebra of Lie) if it has a faithful representation in an associative  $PI$ -algebra. It is easy to show that an  $SPI$ -algebra of Lie is a  $PI$ -algebra of Lie. Examples of  $SPI$ -algebras of Lie are: a subalgebra and a direct sum of  $SPI$ -algebras of Lie, a nilpotent Lie algebra, a solvable Lie algebra of index 2 without center, and a finite-dimensional Lie algebra. A free solvable Lie algebra of index greater than 2 is not an  $SPI$ -algebra of Lie. In Lie algebras there is a known Burnside-type problem, analogous to the problem of A. G. Kurosh in associative algebras: will an algebraic Lie algebra be locally finite? Algebraicity of a Lie algebra is understood here in the sense of Liu Shao-xue<sup>(4)</sup>. The analogous question for algebraic Lie algebras of bounded index was solved only in the special case of Engel

algebras by A. I. Kostrikin <sup>(5)</sup>. In the class of solvable Lie algebras the problem was solved by Liu Shao-xue. In the class of *SPI*-algebras of Lie this problem is solved affirmatively.

**Theorem 2.** An algebraic *SPI*-algebra of Lie is locally finite.

**Lemma 3.** If a Lie algebra is faithfully embeddable in an associative *PI*-algebra that is semisimple in the sense of Jacobson, then the adjoint algebra of the Lie algebra is an associative *PI*-algebra.

Indeed, let  $L$  be embedded in an associative *PI*-algebra  $A$  of degree  $d$ , and suppose  $A$  is semisimple in the sense of Jacobson, i.e.  $A$  is isomorphic to a subdirect sum of primitive algebras  $P_\alpha$ , each of which over its center  $F_\alpha$  has finite dimension  $\leq [d/2]^2$ . Let  $\varphi_\alpha$  denote a homomorphism:

$$A \rightarrow P_\alpha; \quad L^{\varphi_\alpha} \text{ the image of the algebra } L \text{ in } P_\alpha;$$

its linear span  $\overline{L}^{\varphi_\alpha}$  over  $F_\alpha$  has finite dimension  $\leq [d/2]^2$ , and therefore the adjoint algebra  $\mathfrak{A}_\alpha$  of the algebra  $L^{\varphi_\alpha}$  has over  $F_\alpha$  finite dimension  $\leq k = [d/2]^4$ . Consequently, in  $\mathfrak{A}_\alpha$  the standard identity  $S_k(x) = 0$  of degree  $k$  is satisfied. The satisfaction of the identity  $S_k(x) = 0$  in  $\mathfrak{A}_\alpha$  is equivalent to a certain set

$\{f_i(x_{j_1}^{\varphi_\alpha}, \dots, x_{j_t}^{\varphi_\alpha}) = 0\}$ , defining the relations of the algebra  $\overline{L}^{\varphi_\alpha}$ ; here the  $x_{j_s}$  are generators of the algebra  $L$ . Since the coefficients of the identity  $S_k(x) = 0$  lie in the field  $K$ , the  $f_i$  are polynomials in the  $x_{j_s}$  with coefficients in the field  $K$ . In view of the decomposition of  $A$  into the direct sum of the algebras  $P_\alpha$ , in  $L$  there holds a system of defining relations  $\{f_i(x_{j_1}, \dots, x_{j_t}) = 0\}$ , i.e. in the adjoint algebra  $\mathfrak{A}$  of the Lie algebra  $L$  the identity  $S_k(x) = 0$  is satisfied. Lemma 3 is proved.

**Proof of Theorem 2.** Let  $L$  be an algebraic *SPI*-Lie algebra with a finite number of generators  $x_1, x_2, \dots, x_n$  over the field  $K$ ; we shall prove that  $L$  has finite dimension over  $K$ .  $L$  is embedded isomorphically in the associative *PI*-algebra  $A$  generated by the elements  $x_i$ ,  $i = 1, \dots, n$ . Let  $D$  be the Jacobson radical of the algebra  $A$ ;  $D$  is a locally nilpotent algebra <sup>(6,7)</sup>, and therefore  $L_1 = L \cap D$  is locally nilpotent, and the factor-algebra  $\overline{L} = L/L_1$  is embedded isomorphically in the associative *PI*-algebra  $\overline{A} = A/D$ . By Lemma 3, the adjoint algebra  $\mathfrak{A}$  of the Lie algebra  $\overline{L}$  is an associative *PI*-algebra. We shall prove that  $\mathfrak{A}$  has finite dimension over  $K$ . The algebra  $\mathfrak{A}$  has a finite number of generators  $\overline{x}_1, \dots, \overline{x}_n$ ; let  $\overline{\Lambda}$  be the Lie algebra generated by this system of generators in the sense of the multiplication operation  $[a, b] = ab - ba$ . We have:  $\overline{\Lambda} = \overline{L}/\overline{Z}$ , where  $\overline{Z}$  is the center of  $\overline{L}$ . Every element of  $\overline{\Lambda}$  is algebraic in the associative sense. Order the generators  $\overline{x}_i$  arbitrarily; thereby some ordering is defined on the set of all words in the generators  $\overline{x}_i$  <sup>(8)</sup>. We shall show that in the algebra  $\mathfrak{A}$  any word  $v$  of sufficiently large length in the generators  $\overline{x}_i$  has a representation

$$v = v_1 + v_2 + \dots + v_t + \delta,$$

where the words  $v_i$  have the same length and the same composition as  $v$  with respect to the  $\overline{x}_i$ , but are lexicographically smaller;  $\delta$  is a sum of words in the  $\overline{x}_i$

shorter than  $v$ . Indeed,  $\mathfrak{A}$  has bounded height, i.e. there exists a finite number of words  $v_1, v_2, \dots, v_k$  in the  $\bar{x}_i$  and a number  $N$  such that every word  $v$  of length  $\geq N$  in the  $\bar{x}$  has in the algebra  $\mathfrak{A}$  a representation of the form:

$$v = \sum av_{i_1}^{q_1} v_{i_2}^{q_2} \dots v_{i_k}^{q_k}, \quad a \in K, \quad (4)$$

where the words standing under the summation sign in (4) have the same composition and the same length with respect to the  $\bar{x}_i$  as  $v$ , and are lexicographically not greater than  $v$  <sup>(9)</sup>. For every word  $v$  in the  $\bar{x}_i$  one can indicate such a regular word  $t$  in the  $\bar{x}_i$  <sup>(8)</sup> that  $v^{s+1} = at^s b$  for any natural number  $s$ . The word  $t$  uniquely determines an element  $\tau \in \bar{\Lambda}$  <sup>(8)</sup>; from the algebraicity of  $\tau$  in the associative sense it follows that for some  $s$  the word  $w = t^s$  is represented in the form of a linear combination of shorter or lexicographically smaller words. From (4) follows the validity of the last assertion for any sufficiently long word  $v \in \mathfrak{A}$ , and this shows that  $\mathfrak{A}$  has finite dimension. Consequently,  $\bar{\Lambda}$ , and hence also  $\bar{L}$ , has finite dimension. The algebra  $L$ , being an algebraic extension of the locally finite algebra  $\bar{L}$  by means of the locally finite algebra  $L_1$ , also has finite dimension <sup>(4)</sup>, as was required to prove.

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