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Abstract

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ON LINEAR ILL-POSED PROBLEMS

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1. Let X be a linear metric space; Y a Banach space; A a linear continuous operator from X into Y such that A^{-1} exists but is unbounded. Many linear ill-posed problems of mathematical physics and function theory can be reduced to solving the equation

$$Ax = y, \tag{1}$$

where $y \in Y$ is the given element and $x \in X$ the sought element. Because of the unboundedness of the operator A^{-1}, a solution does not exist for all y and is unstable: arbitrarily small variations of y may correspond to arbitrarily large changes in x.

As was shown by A. N. Tikhonov (1), stability can be achieved if the solution is sought in a prescribed compact set $M \subset X$. Put $AM = N$. Then, according to a well-known topological theorem, the mapping $M = A^{-1}N$ is continuous on N. Therefore, if $y \in N$ and, as y varies, we do not leave N, then x will depend continuously on y. The estimate of stability is determined by the modulus of continuity of the mapping A^{-1} on N:

$$\omega(\delta) = \sup \rho(x, x') \quad \text{for } x, x' \in N,\ \|Ax - Ax'\| \leq \delta. \tag{2}$$

If M is symmetric with respect to the zero of the space X, then for sufficiently small δ in (2) one may take $x' = \theta$.

M. M. Lavrent' ev (2) constructed an approximate method for solving equation (1), in which it is assumed that instead of the exact value y its approximation y_δ is known with accuracy δ, and the function $\omega(\delta)$, or its majorant, is known. A similar device is used in the work of F. John (3).

Usually there are no effective criteria that make it possible to establish whether y belongs to N; this has to be assumed known *a priori* (see (1,2)). At the same time, in approximate solution, instead of y one operates with its approximate value y_δ, which may not lie in N, so that $A^{-1}y_\delta$ may not belong to M or may have no meaning.

In connection with this it is natural to change the formulation of the problem and, instead of the exact solution of equation (1), to seek a quasisolution (see below). For the quasisolution the classical well-posedness conditions are preserved (Theorem 1), and to find it one can indicate, by modifying known methods (^{2,5}), convergent processes. If, for the given y, there exists a true solution in M, then the quasisolution coincides with it; in other cases it gives the best approximation to the solution.

2. Definition. We shall call a quasisolution of equation (1) on a given compact set M of the space X, and for a given $y_0 \in N$, a point $x_0 \in M$ for which $\|Ax - y_0\|$ attains its minimum on M.

In application to the Cauchy problem for the Laplace equation, the idea of best approximation is contained in the work of S. N. Mergelyan (^{4}).

Theorem 1. A quasi-solution of equation (1) exists for any nonempty compact set $M \subset X$ and any $y \in Y$. If M is convex and the sphere in the space Y is strictly convex, then the quasi-solution is unique and depends continuously on y.

Proof. Existence follows from the compactness of M. If M is convex, then AM is also convex; $q = Ax$, where x is a quasi-solution, is the point N closest to y (the projection of y onto N).

Under the assumptions of the theorem, the projection of a point onto a convex set is uniquely determined; hence the uniqueness of the quasi-solution. Continuous dependence follows from uniqueness and compactness.

If, under the assumptions of Theorem 1, for a given y_0 equation (1) has in M a true solution x_0, and a sequence $\{y_n\}$ of points of Y converges to y_0, then the sequence of the corresponding quasi-solutions $\{x_n\}$ converges to x_0. Therefore, if y_δ is an approximate value of y_0, then for $y_\delta \to y_0$, for the quasi-solution x_δ we shall have $x_\delta \to x_0$, independently of whether y_δ belongs to N.

If Y is Hilbert and M is convex, then from geometric considerations it follows that if y and y' are elements of Y, and q and q' are their projections onto N, then $\|q - q'\| \leqslant \|y - y'\|$. Therefore, if in Hilbert Y, $\|y - y'\| \leqslant \delta$, and x and x' are the corresponding quasi-solutions, then $\rho(x, x') \leqslant \omega(\delta)$.

In (^{5}) an approximate method for solving equation (1) is constructed under the assumption that $y \in N$. It can be shown that, in the case of linearity of A, for arbitrary y, the approximations constructed there converge to the quasi-solution. Thus (^{5}) gives an approximate method for finding quasi-solutions.

3. More definite results can be obtained by specializing the spaces X and Y. We shall assume that X and Y are Hilbert spaces, $M = \Omega_R$ is the ball $\|x\| \leqslant R$, and A is a completely continuous operator from X into Y. Since the ball is weakly compact in a Hilbert space and this space is metrizable

> with respect to the weak topology, the results of Section 2 for the weak topology in X are applicable to the case under consideration.

Let A^* be the operator adjoint to A. Then A^*A is a self-adjoint, positive, completely continuous operator from Y into Y. Denote by $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant \cdots$ the complete system of its eigenvalues, and by $u_1, u_2, \dots, u_n, \dots$ the complete orthonormal system of its eigenvectors. Let

$$A^*y = \sum_n \beta_n u_n. \tag{3}$$

Theorem 2. The quasi-solution of equation (1) on Ω_R is expressed by the formula

$$x = \sum_n \frac{\beta_n}{\lambda_n + \lambda} u_n, \tag{4}$$

where $\lambda = 0$, if

$$\sum_n \frac{\beta_n^2}{\lambda_n^2} \leqslant R^2; \tag{5}$$

λ is the positive root of the equation

$$\sum_n \frac{\beta_n^2}{(\lambda_n + \lambda)^2} = R^2,$$

if

$$\sum_n \frac{\beta_n^2}{\lambda_n^2} > R^2. \tag{6}$$

Proof. The finding of quasi-solutions on Ω_R reduces to finding in Ω_R a vector x minimizing on Ω_R the quadratic functional $(Ax - y, Ax - y)$. If (5) holds, the minimum is unconditional, and its finding reduces to solving the equation

$$A^*Ax = A^*y. \tag{7}$$

The solution has the form (4) with $\lambda = 0$; it is the quasi-solution (the exact solution).

If (5) is not satisfied, then one must seek the minimum under the condition $(x, x) = R^2$.

Applying the method of Lagrange multipliers, we arrive at an equation of the second kind

$$A^*Ax + \lambda x = A^*y, \tag{8}$$

whose solution gives (4) with $\lambda > 0$.

When y is given approximately, owing to errors, as a rule, (6) will hold. The presence of positive λ in the denominator ensures strong convergence of the series (4).

The method of Theorem 2 is a development of the device of M. M. Lavrent' ev (see (^{2})); however, unlike the latter, for the application of this method there is no need to know the function $\omega(\delta)$ defined by (2). This makes it possible to restrict oneself to weak compactness of the set M.

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Note: Figure translations are in progress. See original paper for figures.

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