MATHEMATICS

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Abstract

Full Text

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SPECIAL MODULES

AND SPECIAL RADICALS

As is known (^{1,2}), the Jacobson and Baer radicals of an arbitrary associative ring are naturally considered from the point of view of representation theory, or, what is the same thing, from the point of view of module theory. In such an exposition, an essential role is played, respectively, by irreducible and primary modules.

In the present note we show that any special radical (^{3}) can also be considered from the same point of view. The role of irreducible and primary modules will be played by the special modules that generalize these concepts, the definition of which will be given below.

Let A be an arbitrary associative ring. The annihilator (right) of an A-module $\mathfrak{M}$ is the set

$$(0:\mathfrak{M})_A = \{a \mid a \in A,\ \mathfrak{M}a = 0\}.$$

If $(0:\mathfrak{M})_A = 0$, then the A-module $\mathfrak{M}$ is called **faithful**.

An A-module $\mathfrak{M}$ is called a **primary** A-module if: 1) $\mathfrak{M}A = \{\sum x_i a_i \mid x_i \in \mathfrak{M},\ a_i \in A\} \neq 0$; 2) from the equality $xB = 0$, where $x \in \mathfrak{M}$, B is an ideal in A, it follows that $x = 0$ or $B \subseteq (0:\mathfrak{M})_A$.

Associate with each ring A some class Σ_A of A-modules, and let

$$\Sigma = \bigcup_A \Sigma_A.$$

Definition 1. The class Σ is called a **special class of modules** if and only if the following conditions are satisfied:

I. If $\mathfrak{M} \in \Sigma_A$, then $\mathfrak{M}$ is a primary A-module.

II. If $\mathfrak{M} \in \Sigma_B$ and B is an ideal of the ring A, then $\mathfrak{M}B \in \Sigma_A$.

III. If $\mathfrak{M} \in \Sigma_A$ and B is such an ideal of the ring A that $\mathfrak{M}B \neq 0$, then $\mathfrak{M} \in \Sigma_B$.

IV. If $\mathfrak{M} \in \Sigma_A$, $\bar{A} = A/P$, $P \subseteq (0:\mathfrak{M})_A$, then, under the composition $x\bar{a} = xa$, where $x \in \mathfrak{M}$, $a \in A$, $\bar{a} \in \bar{A}$, we obtain $\mathfrak{M} \in \Sigma_{\bar{A}}$.

Conversely: if $\mathfrak{M} \in \Sigma_{\bar{A}}$, where $\bar{A} = A/P$ is an arbitrary homomorphic image of the ring A, then under the composition $xa = x\bar{a}$ we obtain $\mathfrak{M} \in \Sigma_A$.

Every A-module $\mathfrak{M} \in \Sigma_A$, where Σ is a special class of modules, will be called a **special A-module**.

Definition 2. Let Σ be a special class of modules and A an arbitrary associative ring. The **Σ-radical** of the ring A is the intersection

$$R(\Sigma, A) = \bigcap_{\mathfrak{M} \in \Sigma_A} (0 : \mathfrak{M})_A.$$

If the set Σ_A is empty, then we shall assume that $R(\Sigma, A) = A$, and call the ring A **Σ-radical**. If $R(\Sigma, A) = 0$, then we shall call the ring A **Σ-semisimple**. Finally, an ideal B of the ring A will be called a **Σ-radical ideal** if B, considered as a ring, is a Σ-radical ring.

Proposition 1. Every homomorphic image of a Σ-radical ring is a Σ-radical ring.

Proposition 2. The factor ring $A/R(\Sigma, A)$ is Σ-semisimple.

Theorem 1. If B is an ideal of the ring A, then the equality

$$R(\Sigma, B) = B \cap R(\Sigma, A)$$

holds.

Corollary 1. $R(\Sigma, A)$ is a Σ-radical ideal of the ring A.

Corollary 2. Every ideal of a Σ-semisimple ring is a Σ-semisimple ring.

Corollary 3. Every ideal of a Σ-radical ring is a Σ-radical ring.

Proposition 3. The Σ-radical of a ring is the sum of all its Σ-radical ideals.

It follows from Propositions 1, 2, 3 that the Σ-radical of a ring is a radical in the sense of A. G. Kurosh (^{4}).

Proposition 4. Σ-semisimple rings are characterized by the fact that they are subdirect sums of rings A_α possessing faithful special A_α-modules.

Recall that a nonzero ring is called **prime** if the product of any two nonzero ideals is a nonzero ideal. The class of prime rings M is called a **special class of rings** (^{3}) if: 1) every nonzero ideal of a ring in M belongs to the class M; 2) every prime ring containing, as a nonzero ideal, a ring from M, itself belongs to the class M.

We shall say that a ring A belongs to the **Σ-special class of rings** $M(\Sigma)$ if and only if there exists a faithful special A-module $\mathfrak{M}$, i.e. $\mathfrak{M} \in \Sigma_A$.

Theorem 2. Every Σ-special class of rings $M(\Sigma)$ is a special class of rings. Conversely: every special class of rings will be Σ-special for some special class Σ of modules.

By an M-**special radical** we shall mean the upper radical in the sense of A. G. Kurosh (^{4}), determined by a special class of rings M. Recall that if R is an M-special radical, then R-radical rings are characterized by the fact that they are not mapped homomorphically onto rings from the class M. An M-**special ideal** of a ring A is an ideal P such that the factor ring A/P belongs to the class M. Recall that the M-special radical of an arbitrary associative ring A is the intersection of all its M-special ideals.

Proposition 5. An ideal P of a ring A is $M(\Sigma)$-special if and only if

$$P = (0 : \mathfrak{M})_A$$

for a suitable $\mathfrak{M}$ from Σ_A.

Corollary 4. The Σ-radical $R(\Sigma, A)$ of an arbitrary associative ring A is the intersection of all its $M(\Sigma)$-special ideals, i.e. $R(\Sigma, A)$ is the $M(\Sigma)$-special radical of the ring A.

Let us give several examples of special classes of modules.

1. The class Σ of all prime modules, i.e. the class

   $$\Sigma = \bigcup_A \Sigma_A,$$

   where Σ_A is the class of all prime A-modules, is a special class of modules. The corresponding Σ-special class of rings $M(\Sigma)$ coincides with the class of all prime rings. The $M(\Sigma)$-special ideals in this case will be prime ideals. By virtue of Corollary 4, the Σ-radical $R(\Sigma, A)$ of the ring A is the intersection of all its prime ideals, i.e. $R(\Sigma, A)$ coincides with the Baer-McCoy radical (^{1}).

2. Recall that an A-module $\mathfrak{M}$ is called **irreducible** if $\mathfrak{M}A \neq 0$ and its only submodules are 0 and $\mathfrak{M}$. The class Σ of all irreducible modules is a special class of modules. The corresponding Σ-special class of rings $M(\Sigma)$ coincides with the class of all Jacobson primitive rings. The $M(\Sigma)$-special ideals are the primitive ideals. According to Corollary 4, the Σ-radical $R(\Sigma, A)$ of the ring A is the intersection of all its primitive ideals, i.e. $R(\Sigma, A)$ is the Jacobson radical (^{1}).

3. We shall call a primary A-module $\mathfrak{M}$ antiregular if there exists a nonzero element $a \in A$ such that from $a \in B$, where B is an ideal of the ring A, it follows that $B \subseteq (0 : \mathfrak{M})_A$.

The class Σ of all antiregular modules is a special class of modules. The corresponding Σ-special class of rings $M(\Sigma)$ will be the class of all directly indecomposable rings with idempotent heart. The $M(\Sigma)$-special ideals in this case will be the simple submaximal ideals. The Σ-radical $R(\Sigma, A)$ of the ring A is the intersection of all its simple submaximal ideals, i.e. $R(\Sigma, A)$ is the antiregular radical of the ring (^{5}).

4. We shall call an A-module $\mathfrak{M}$ simple if $\mathfrak{M}A \neq 0$ and in every ideal B of the ring A for which $\mathfrak{M}B \neq 0$, there exists an element $b \in B$ such that $xb = x$ for every $x \in \mathfrak{M}$. The class Σ of all simple modules is a special class of modules. The corresponding Σ-special class of rings $M(\Sigma)$ coincides with the class of simple rings with identity. The $M(\Sigma)$-special ideals are those ideals whose factor rings are simple rings with identity, i.e. the maximal modular ideals. The Σ-radical $R(\Sigma, A)$ will be the intersection of all modular maximal ideals of the ring A, i.e. $R(\Sigma, A)$ is the Brown-McCoy radical ([6]).
5. We shall call an A-module $\mathfrak{M}$ Boolean if $\mathfrak{M}A \neq 0$ and for every ideal B of the ring A such that $\mathfrak{M}B \neq 0$, the condition is fulfilled: $xb = x$ for every $x \in \mathfrak{M}$ and every nonzero element $b \in B$. The class of all Boolean modules is a special class of modules. The Σ-special class of rings $M(\Sigma)$ is the field with two elements. According to Corollary 4, the Σ-radical $R(\Sigma, A)$ of an arbitrary associative ring A is the intersection of all its ideals such that the factor rings by them are fields with two elements, i.e. $R(\Sigma, A) = R_M$ ([5]).
6. We shall call an A-module $\mathfrak{M}$ an A-module without zero divisors if $\mathfrak{M}A \neq 0$ and from $xa = 0$, where $x \in \mathfrak{M}$, $a \in A$, it follows that either $x = 0$ or $a \in (0 : \mathfrak{M})_A$. The class of all modules without zero divisors is a special class of modules. The Σ-special class of rings coincides with the class of all rings without zero divisors. The $M(\Sigma)$-special ideals are those ideals whose factor rings are rings without zero divisors, i.e. the completely prime ideals. The Σ-radical $R(\Sigma, A)$ is the intersection of all completely prime ideals of the ring A, i.e. $R(\Sigma, A)$ is the generalized nil-radical (the compressive radical) ([5,7]).
7. We shall call a primary A-module $\mathfrak{M}$ a Köthe A-module if the factor ring $A/(0 : \mathfrak{M})_A$ contains no nonzero nil-ideals. The class of all Köthe modules is a special class of modules. The Σ-special class of rings $M(\Sigma)$ coincides with the class of all primary rings without nonzero nil-ideals. The Σ-radical $R(\Sigma, A)$ is the intersection of all such ideals whose factor rings are primary rings without nonzero nil-ideals, i.e. it is the Köthe radical ([8]).
8. We shall call a primary A-module $\mathfrak{M}$ a Levitzki A-module if the factor ring $A/(0 : \mathfrak{M})_A$ is a ring without locally nilpotent ideals. The class of all Levitzki modules is a special class of modules. The Σ-special class of rings $M(\Sigma)$ coincides with the class of all primary rings without locally nilpotent ideals. The Σ-radical $R(\Sigma, A)$ is the intersection of all such ideals of the ring A whose factor rings are primary rings without locally nilpotent ideals, i.e. it is the Levitzki radical ([1,9,10]).
9. The class Σ of all irreducible modules without zero divisors is a special class of modules. The corresponding Σ-special class of rings $M(\Sigma)$ will be the class of all division rings. According to Corollary 4, the Σ-radical $R(\Sigma, A)$ is the intersection of all such ideals of the ring A whose factor rings

are division rings, i.e. $R(\Sigma, A)$ is the radical (the division-ring radical) R_4 ([5,11]).

Recall that an A-module $\mathfrak{M}$ is called **interversible** (11) if, for any $x \in \mathfrak{M}$, $a, b \in A$, the equality $xabA = xbaA$ holds.

Proposition 6. The class of irreducible interversible modules coincides with the class of irreducible modules without zero divisors.

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