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Abstract

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ON BICOMPACT GROUPS WITH THE MAXIMALITY CONDITION FOR SUBGROUPS

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Adhering to the terminology adopted in the book of L. S. Pontryagin (^{1}), we shall say that a topological group G satisfies the maximality condition for subgroups if every increasing chain of its subgroups terminates. If every increasing chain of its abelian subgroups terminates, then it is said to satisfy the maximality condition for abelian subgroups.

A. I. Mal' tsev (^{3}) proved that a discrete group G satisfying the maximality condition for abelian subgroups also satisfies the maximality condition for subgroups, if it is a group of one of the following two types: a) G is a solvable group, b) G is a locally nilpotent group. These assertions can be extended to other classes of topological groups.

For example, the following theorems hold:

Theorem 1. *A bicompact solvable group G satisfying the maximality condition for abelian subgroups satisfies the maximality condition for subgroups.*

Theorem 2. *A bicompact locally nilpotent group G satisfying the maximality condition for abelian subgroups satisfies the maximality condition for subgroups.*

Here we shall give a complete proof of the first of these theorems. For this, two auxiliary propositions will be needed:

Lemma 1. *A bicompact metabelian group satisfies the maximality condition for subgroups if it possesses a maximal abelian subgroup with the maximality condition for subgroups.*

We omit the proof, since it almost literally repeats the proof of Theorem 3 from the paper of N. F. Sesekin (^{4}).

Lemma 2. *Let Γ be a bicompact solvable group of automorphisms of a bicompact abelian group G with the maximality condition for subgroups. Then Γ satisfies the maximality condition for subgroups.*

Proof. The group G, satisfying the conditions of the lemma, decomposes into the direct product of a finite subgroup M and a finite number of subgroups isomorphic to the additive group J_p^+ of the ring J_p of integral p-adic numbers for certain prime numbers p. Let Φ be the set of all automorphisms in Γ that leave fixed all elements of M. Then Φ is a bicompact group of automorphisms of the group $\Gamma = G/M$. Therefore it is enough to prove the lemma for the group F and the group Φ of its automorphisms. The group Γ decomposes into the direct product $P \times P' \times \cdots$ of a finite number of Sylow subgroups $P, P', \ldots$ for distinct prime numbers $p, p', \ldots$. If Φ induces the group of automorphisms Δ' of the group P', etc., then it is easy to see that $\Phi \subseteq \Delta \times \Delta' \times \cdots$. Therefore, to prove the lemma it is enough to show that each factor $\Delta, \Delta', \ldots$ of the product satisfies the maximality condition for subgroups. Let us prove this, for example, for Δ. Each automorphism from this group Δ is represented by a matrix over the ring J_p. In view of the theorem of A. I. Mal' tsev-

$\cdots$(3) on soluble groups of matrices, in Δ there is an algebraic invariant subgroup ψ of finite index with the following property: there exists a matrix X, with entries from the algebraic closure of the field K of p-adic numbers, such that for any matrix A from ψ all similar matrices $A' = X^{-1}AX$ have triangular form:

$$A' = \left\| \begin{array}{ccccc} \lambda_1 & \alpha_{12} & \cdot & \cdot & \alpha_{1n} \\ 0 & \lambda_2 & \cdot & \cdot & \alpha_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \lambda_n \end{array} \right\|.$$

The closure $\bar{\psi}$ of the algebraic subgroup ψ in Δ also consists of matrices reducible to triangular form by means of the same matrix X. Therefore one may assume that $\bar{\psi}$ has been chosen as ψ, i.e. that ψ is closed in Δ. Denote by Σ the field obtained from the field K by adjoining all elements of the matrix X. Then $\lambda_i \in \Sigma$ $(i = 1, 2, \ldots, n)$, and all λ_i with fixed index i constitute a subgroup Λ_i of the multiplicative group Λ of all p-adic units of the field Σ. The group Λ is bicompact and satisfies the maximality condition for subgroups (see (2), § 15). The homomorphic mapping $A \to \lambda_i$ of the group ψ in Δ has the closed image Λ_i. If Ω_i is the kernel of this homomorphism, then $\psi/\Omega_i \simeq \Lambda_i$. Let $\Omega = \Omega_1 \cap \Omega_2 \cap \cdots \cap \Omega_n$. Then ψ/Ω is isomorphic to a subgroup of the group $\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n$. It follows that the factor group ψ/Ω satisfies the maximality condition for subgroups. But the subgroup Ω also satisfies this condition, since it is isomorphic to a subgroup of the group of all triangular matrices with 1 on the main diagonal and with entries from the ring J_p of integral p-adic numbers. Therefore the group Δ also satisfies this condition.

Proof of Theorem 1 (see Theorem 8 in (3)). Let the group G satisfy the conditions of this theorem. Then it has a finite invariant series of closed subgroups with abelian factors:

$$1 = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_m = G. \tag{1}$$

First consider the case when its length m is equal to 2. Then the series (1) has the form $1 \subset G_1 \subset G$. If H is the centralizer of G_1 in G, then $G_1 \subset$

H, and H is invariant in G. From Lemma 1 it follows that H satisfies the maximality condition for subgroups. The group $\Gamma = G/H$ is isomorphic to a group of automorphisms of the abelian group G_1, which satisfies the maximality condition for subgroups. But then, in view of Lemma 2, the group Γ satisfies the maximality condition. Therefore the group G also satisfies this condition. Now let $m > 2$, and suppose that for groups possessing series of the form (1) of length $< m$, the theorem is true. We shall prove its validity for the group G. In view of the case $m = 2$ considered above, all abelian subgroups of the factor group $\bar{G} = G/G_1$ satisfy the maximality condition for subgroups. Therefore, by assumption, the group $\bar{G}$ satisfies the maximality condition for subgroups. Since G_1 and $\bar{G}$ are bicompact, it follows that the group G also satisfies this condition.

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Note: Figure translations are in progress. See original paper for figures.

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