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Fig. 1. Projection onto the plane $z = z_0$

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Abstract

Full Text

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ELECTROMAGNETIC FIELD EXCITED BY AN ELECTRIC DIPOLE IN A WEDGE-SHAPED REGION*

(Presented by Academician V. A. Fock on 17 IV 1962)

The solution S of the scalar problem of diffraction of a spherical wave in an angular region in the form of the integral

$$S(\rho, \varphi, z, \rho_0, \varphi_0, z_0) = \frac{1}{2\pi i} \int_{\gamma+\varphi} \frac{e^{ikR(\varphi-\alpha)}}{R(\varphi-\alpha)} s(\alpha)\, d\alpha, \tag{1}$$

where

$$R(\varphi - \varphi_0) = \sqrt{\rho^2 - 2\rho\rho_0 \cos(\varphi - \varphi_0) + \rho_0^2 + (z - z_0)^2},$$

was obtained by Carslaw (^{1}) for the boundary conditions $\partial S/\partial n = 0$ or $S = 0$, when the corresponding functions $s(\alpha)$ can be represented in the form

$$s(\alpha) = (\pi/4\Phi)\,[\operatorname{ctg}(\pi/4\Phi)(\alpha - \varphi_0) \pm \operatorname{ctg}(\pi/4\Phi)(\alpha - 2\Phi + \varphi_0)].$$

Subsequently (^{4}) a method was proposed for setting up and solving a functional equation for the function $s(\alpha)$ such that the integral (1)** satisfied the conditions of one or another diffraction problem whose solution is representable in the form (1).

In the present paper a generalization of this method to the case of vector functions S and s is proposed. The method is demonstrated on the example of a problem, previously considered by another method (^{2}), concerning the field of an electric dipole in an angular region with ideally conducting faces. For simplicity we formulate the problem in terms of the electric Hertz vector.

Fig. 2. The β-plane

Figure 2: Fig. 2. The β-plane

Fig. 1. Projection onto the plane $z = z_0$

Let, in cylindrical coordinates ρ, φ, z, the angular region be defined by the inequalities $\rho > 0$, $-\Phi < \varphi < \Phi$, $-\infty < z < \infty$. We shall characterize by the radius vectors $\mathbf{r}$ and $\mathbf{r}_0$ an arbitrary point ρ, φ, z and a fixed point ρ_0, φ_0, z_0 ($\rho_0 > 0$, $|\varphi_0| < \Phi - \varepsilon$), at which the moment vector of an electric dipole is specified,

$$\mathbf{P} = P(\mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta).$$

Of the orthogonal unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, the first is directed along the z-axis behind the plane of the drawing (see Fig. 1), i.e. along the edge of the wedge; the second lies in the plane $z = z_0$ and makes the prescribed angle ψ with the plane $\varphi = 0$. The third is defined as

$$\mathbf{e}_3 = [\mathbf{e}_1, \mathbf{e}_2].$$

The required Hertz vector $\vec{\Pi}(\mathbf{r})$ satisfies, inside the region, the Helmholtz equation

$$(\Delta + k^2)\vec{\Pi} = -4\pi \mathbf{P}\, \delta(\mathbf{r} - \mathbf{r}_0) \tag{2}$$

and is an analytic function of the parameters k and Φ***. On the ideally conducting faces $\varphi = \pm\Phi$ ($\mathbf{n}_{\pm\Phi}$ are the vectors of the normals) the Hertz vector satisfies the boundary conditions

$$[\mathbf{n}_{\pm\Phi}, \vec{\Pi}] = 0; \qquad \partial(\mathbf{n}_{\pm\Phi}, \vec{\Pi})/\partial n_{\pm\Phi} = 0. \tag{3*}$$

Instead of these conditions one may use the principle of symmetry* with respect to the faces $\varphi = \pm\Phi$ of the field $\vec{\Pi} = \vec{\Pi}(\varphi)$, analytically continued through them as a solution of the equation $(\Delta + k^2)\vec{\Pi} = 0$. In the present case this principle is expressed by four equalities, valid for arbitrary values of the angle $\varphi'_{\pm\Phi} = \mp\Phi + \varphi$, measured from the considered face of the wedge $\varphi = \pm\Phi$:

Fig. 2. The β-plane

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* Report at the Odessa Symposium on Wave Diffraction, 27 IX 1960.

** In the general case the integral (1) is called in (^{4}) a Sommerfeld integral with one line of branching ($\rho = 0$) and with a kernel in the form of a spherical wave.

*** Analyticity of the dependence on k is a necessary condition in order that the function $\vec{\pi}(\mathbf{r}, k)$ be a forced oscillation (^{5,6}).

$$[\mathbf{n}_{\pm\Phi};\ \vec{\Pi}(\pm\Phi+\varphi'_{\pm\Phi})+\vec{\Pi}(\pm\Phi-\varphi'_{\pm\Phi})]=0;$$

$$(\mathbf{n}_{\pm\Phi};\ \vec{\Pi}(\pm\Phi+\varphi'_{\pm\Phi})-\vec{\Pi}(\pm\Phi-\varphi'_{\pm\Phi}))=0. \tag{3}$$

The condition that there be no sources at the edge itself $\rho=0$ and at infinitely remote points $R(\varphi-\varphi_0)\to\infty$ may be formulated by using the analyticity and single-valuedness of the dependence of the desired function $\vec{\Pi}(\mathbf{r};k;\Phi)$ on the parameters k and Φ, as well as the principle of limit absorption ($^{4-7}$). Namely, we require that, for $0<\Phi<\pi/2$ and values of k lying in a neighborhood of at least one complex k_1 ($\operatorname{Im} k_1>0$), first,

$$|\vec{\Pi}|\to 0 \quad \text{as } \rho\to 0$$

uniformly with respect to φ, and, second, throughout the region**

$$\sup\left|\vec{\Pi}-\{\mathbf{P}\exp[ikR(\varphi-\varphi_0)]\}/R(\varphi-\varphi_0)\right|<\infty. \tag{4}$$

Conditions (2), (3), (4) constitute the mathematical formulation of the problem of forced oscillations*** of the electromagnetic field excited by an electric dipole in an angular region.

We shall show that it is possible to find, in a definite way, such a vector function of the complex variable $\vec{\pi}(\alpha)$ that the desired solution $\vec{\Pi}(\mathbf{r})$ is represented by a Sommerfeld integral with a kernel in the form of a spherical wave****

$$\vec{\Pi}(\mathbf{r})=\frac{1}{2\pi i}\int_{\gamma+\varphi}\frac{e^{ikR(\varphi-\alpha)}}{R(\varphi-\alpha)}\vec{\pi}(\alpha)\,d\alpha=\frac{1}{2\pi i}\int_{\gamma}\frac{e^{ikR(\beta)}}{R(\beta)}\vec{\pi}(\beta+\varphi)\,d\beta, \tag{5}$$

where γ is the point on the contour of integration, shown in the form of two loops on the complex β-plane in Fig. 2. The branch points $\beta=2\pi n\pm ic$, $c=\operatorname{Ar\,ch}\{[\rho^2+\rho_0^2+(z-z_0)^2]/2\rho\rho_0\}$, with vertical cuts going to infinity, are also shown there. On the upper Riemann sheet the regions depending on $\delta=2\arg k$, where $\operatorname{Re}[ikR(\beta)]>0$, are shaded.

Before determining, by means of the symmetry principle (3), the concrete form of the function $\vec{\pi}(\alpha)$, let us impose on this function general requirements sufficient for satisfying all the remaining requirements of the problem. Let, for any complex α, the function $\vec{\pi}(\alpha)=\vec{\pi}(\alpha;\Phi)$ be a single-valued analytic function of the parameter Φ and not depend on the parameter k.

With respect to the dependence on the variable α, impose on $\vec{\pi}(\alpha)$ the following requirements: 1) for all $\Phi>0$, $\vec{\pi}(\alpha)$ is a meromorphic function, having singularities only on the real axis and increasing as $\operatorname{Im}\alpha\to\pm\infty$

not faster than some exponential; 2) the difference $\vec{\pi}(\alpha) - \mathbf{P}/(\alpha - \varphi_0)$ has no singularities in the strip $|\operatorname{Re}\alpha| < \Phi + \varepsilon$; 3) for $0 < \Phi < \pi/2$ and $\operatorname{Im}\alpha \to \pm\infty$ the function $\vec{\pi}(\alpha) \to 0$.

The listed requirements ensure the analyticity of the desired solution (5) with respect to k, as well as satisfaction of the Helmholtz equation (2) and of the radiation condition (4). Indeed, it follows from 1) that, in the interval $0 \leqslant \arg k \leqslant \pi/2$, the integral (5) is a single-valued analytic function of the parameter k, since the kernel $\{\exp[ikR(\varphi-\alpha)]\}/R(\varphi-\alpha)$ is such a function and the integral converges uniformly. Hence it also follows that, in the indicated interval, the integral (5) remains bounded as $\rho \to \infty$.

It follows from (2) that the integral (5) in the angular region is a solution of equation (2). Indeed, let $0 \leqslant \pi/2 - \arg k < \varepsilon/2$. Then, deforming the contour of integration into two vertical straight lines, using condition (2) and Cauchy' s theorem, we can rewrite (5) in the form

$$\vec{\Pi}(\mathbf{r}) = \mathbf{P}\,\frac{e^{ikR(\varphi-\varphi_0)}}{R(\varphi-\varphi_0)} + \frac{1}{2\pi i}\left(\int_{-i\infty-(\Phi+\varepsilon)}^{i\infty-(\Phi+\varepsilon)} - \int_{-i\infty+(\Phi+\varepsilon)}^{i\infty+(\Phi+\varepsilon)}\right)\frac{e^{ikR(\varphi-\alpha)}}{R(\varphi-\alpha)}\,\vec{\pi}(\alpha)\,d\alpha, \quad (6)$$

where the first term on the right-hand side satisfies the inhomogeneous equation (2) in all space, while the second satisfies the corresponding homogeneous equation in the angular region. Since equation (2) is satisfied for values of $\pi/2 - \arg k$ close to zero, by virtue of analyticity it is also satisfied for all k.

From 2) and 3) there follows fulfillment of the radiation condition (4). Indeed, the integral entering the right-hand side of (6) represents precisely the function that must be uniformly bounded for $0 < \Phi < \pi/2$ according to (4). For $\rho > 0$ this integral is bounded for all $\Phi > 0$. But for $0 < \Phi < \pi/2$, in the limit $\rho = 0$, it reduces to the residue

$$-\frac{\exp\left[ik\sqrt{\rho_0^2 + (z - z_0)^2}\right]}{\sqrt{\rho_0^2 + (z - z_0)^2}},$$

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* The principle of symmetry was used earlier (^{4}) for formulating problems on a scalar field in regions of arbitrary form.

** The boundedness condition in the electromagnetic problem, imposed for purely imaginary values of the parameter k, was proposed by V. A. Fock (^{3}).

*** For the scalar case, the general formulation of the problem of forced oscillations in an arbitrary region and the uniqueness theorems are considered in (^{4,5}).

**** The Sommerfeld integral with a kernel in the form of a plane wave has been studied in more detail (^{4,8}).

which also represents a finite quantity*. Consequently, for $0 < \Phi < \pi/2$ (in a neighborhood of positive imaginary values of the parameter k) the condition of uniform boundedness (4) of this integral is satisfied.

Let us now turn to finding the form of the function $\vec{\pi}(\alpha)$ by means of the symmetry principle (3). Substituting (5) into (3), we find

$$\int_{\gamma+\varphi'_{\pm\Phi}} \frac{e^{ikR(\varphi'_{\pm\Phi}-\alpha)}}{R(\varphi'_{\pm\Phi}-\alpha)} \left[\mathbf{n}_{\pm\Phi};\, \vec{\pi}(\alpha\pm\Phi) - \vec{\pi}(-\alpha\pm\Phi)\right] d\alpha = 0,$$

$$\int_{\gamma+\varphi'_{\pm\Phi}} \frac{e^{ikR(\varphi'_{\pm\Phi}-\alpha)}}{R(\varphi'_{\pm\Phi}-\alpha)} \left(\mathbf{n}_{\pm\Phi};\, \vec{\pi}(\alpha\pm\Phi) + \vec{\pi}(-\alpha\pm\Phi)\right) d\alpha = 0.$$

(We note in passing that substituting (5) into the boundary conditions (3^*) leads to these same equalities with $\varphi'_{\pm\Phi} = 0$.) To satisfy these equalities it is sufficient to require fulfillment of four functional equations

$$[\mathbf{n}_{\pm\Phi};\, \vec{\pi}(\alpha\pm\Phi) - \vec{\pi}(-\alpha\pm\Phi)] = 0;$$

$$(\mathbf{n}_{\pm\Phi};\, \vec{\pi}(\alpha\pm\Phi) + \vec{\pi}(-\alpha\pm\Phi)) = 0, \tag{7}$$

from which, according to 2) and 3), one must find the function $\vec{\pi}(\alpha)$, regular in the strip $|\operatorname{Re}\alpha| < \Phi + \varepsilon$ everywhere except for the simple pole $\alpha = \varphi_0$ with residue $\mathbf{P}$, and tending to zero in this strip for $0 < \Phi < \pi/2$ as $\operatorname{Im}\alpha \to \pm\infty$.

* In this case, according to (6), $\vec{\pi} = 0$.

Assuming temporarily that $0 < \Phi < \pi/2$, $\varphi_0 > 0$, and representing $\vec{\pi}(\alpha)$ in the strip $|\operatorname{Re}\alpha| < \Phi + \varepsilon$ by Fourier integrals (^{4,9})

$$\vec{\pi}(\alpha) = \frac{i}{\sqrt{2\pi}} \int_{-i\infty}^{i\infty} F(w) e^{-iw\alpha}\, dw \quad (\operatorname{Re}\alpha < \varphi_0);$$

$$\vec{\pi}(\alpha) = \frac{i}{\sqrt{2\pi}} \int_{-i\infty}^{i\infty} \left[F(w) - P\sqrt{2\pi}\, e^{iw\varphi_0}\right] e^{-iw\alpha}\, dw \quad (\operatorname{Re}\alpha > \varphi_0),$$

we obtain from (7) the system of equations

$$[\mathbf{n}_{\Phi};\, F(w)e^{-iw\Phi} - F(-w)e^{iw\Phi}] = -2\sqrt{2\pi}\, i\, [\mathbf{n}_{\Phi} P] \sin w(\Phi - \varphi_0);$$

$$[\mathbf{n}_{-\Phi};\, F(w)e^{iw\Phi} - F(-w)e^{-iw\Phi}] = 0,$$

$$(\mathbf{n}_\Phi;\ F(w)e^{-iw\Phi} + F(-w)e^{iw\Phi}) = 2\sqrt{2\pi}\,(\mathbf{n}_\Phi P)\cos w(\Phi - \varphi_0);$$

$$(\mathbf{n}_{-\Phi};\ F(w)e^{iw\Phi} + F(-w)e^{-iw\Phi}) = 0,$$

solving which and evaluating the Fourier integrals, we find

$$\vec{\pi}(\alpha) = \frac{\pi P}{4\Phi}\left\{\mathbf{e}_{\alpha-\varphi_0}\operatorname{ctg}\frac{\pi}{4\Phi}(\alpha - \varphi_0) + [\mathbf{e}_{\alpha-\varphi_0}, \bar{\mathbf{e}}_1] - \mathbf{e}_{\alpha+\varphi_0-2\psi}\operatorname{ctg}\frac{\pi}{4\Phi}(\alpha - 2\Phi + \varphi_0) - [\mathbf{e}_{\alpha+\varphi_0-2\psi}, \mathbf{e}_1]\right\}. \tag{8}$$

Here and below we have used the notation $\mathbf{e}_\alpha = \mathbf{e}_1\cos\theta + (\mathbf{e}_2\cos\alpha + \mathbf{e}_3\sin\alpha)\sin\theta$.

Expression (8), obtained under the assumption $0 < \Phi < \pi/2$, directly realizes the analytic continuation to all positive values of the parameter Φ. For $\Phi > \pi/2$ the function $\vec{\pi}(\alpha)$ grows as $\operatorname{Im}\alpha \to \pm\infty$ in such a way that $\vec{\pi}(\alpha) = O\{\exp[(1 - \pi/2\Phi)|\operatorname{Im}\alpha|]\}$. Consequently, condition 1) is satisfied.

Substitution of (8) into (5) gives an exact expression in the form of a Sommerfeld integral for the Hertz vector $\vec{\Pi}(\mathbf{r})$, excited by an electric dipole in an angular domain.

The corresponding expressions for the magnetic field $\mathbf{H} = -ik\operatorname{rot}\vec{\Pi}$ and the electric field $\mathbf{E} = \operatorname{grad}\operatorname{div}\vec{\Pi} + k^2\vec{\Pi}$ are also obtained by substituting (8) into the formulas

$$\mathbf{H}(\mathbf{r}) = \frac{1}{2\pi i}\int_\gamma \frac{e^{ikR}}{R}\left(k^2 + \frac{ik}{R}\right)\left[\frac{\mathbf{r} - \mathbf{r}_0(\beta+\varphi)}{R}, \vec{\pi}(\beta+\varphi)\right] d\beta;$$

$$\mathbf{E} = \frac{1}{2\pi i}\int_\gamma \frac{e^{ikR}}{R}\left\{\left(-k^2 - \frac{3ik}{R} + \frac{3}{R^2}\right)\left(\frac{\mathbf{r} - \mathbf{r}_0(\beta+\varphi)}{R}, \vec{\pi}(\beta+\varphi)\right)\frac{\mathbf{r} - \mathbf{r}_0(\beta+\varphi)}{R} + \left(k^2 + \frac{ik}{R} - \frac{1}{R^2}\right)\vec{\pi}(\beta+\varphi)\right.$$

where, for brevity, $R = R(\beta)$ and $\mathbf{r}_0(\alpha) = z_0\mathbf{e}_1 + \rho_0[\mathbf{e}_2\cos(\alpha-\psi) + \mathbf{e}_3\sin(\alpha-\psi)]$.

It is easy to verify that as $\rho \to 0$ the functions $\vec{\Pi}, \mathbf{E}, \mathbf{H}$ are bounded if $0 < \Phi < \pi/2$, and have order of growth $\rho^{\pi/2\Phi - 1}$ if $\Phi > \pi/2$.

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Note: Figure translations are in progress. See original paper for figures.

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