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Abstract

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MATHEMATICS

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ON A CLASS OF MARKOV PROCESSES WHOSE EXIT PROBABILITIES ARE MAJORIZED BY THE EXIT PROBABILITIES OF A WIENER PROCESS

(Presented by Academician P. S. Aleksandrov, 14 V 1962)

In the l-dimensional Euclidean space R^l let us consider the Wiener process $\hat{X} = (\hat{x}_t, \hat{\mathcal{M}}_t, \hat{P}_x)$ and a strictly Markov process $X = (x_t, \xi, \mathcal{M}_t, P_x)$, obtained by a random change of time in some subprocess of the process $\hat{X}$.* Denote by τ_U (respectively, $\hat{\tau}_U$) the moment of first exit of the process X (the process $\hat{X}$) from the domain U**, and put

$$\pi_U(x, \Gamma) = P_x\{x(\tau_U) \in \Gamma\}, \qquad \hat{\pi}_U(x, \Gamma) = \hat{P}_x\{\hat{x}(\hat{\tau}_U) \in \Gamma\}$$

for any Borel set Γ in R^l. The system of measures $\pi_U(x, \Gamma)$, evidently, satisfies the condition:

A. $\pi_U(x, \Gamma) \leq \hat{\pi}_U(x, \Gamma)$ for all Borel Γ.

If, moreover, $P_x\{\xi > 0\} = 1$ for all $x \in R^l$, then the following condition also holds:

B. $\pi_{U_n}(x, R^l) \to 1$ as $n \to \infty$ for any sequence of domains U_n, each of which contains the point x and whose diameters tend to zero as $n \to \infty$.

This assertion admits a converse. More precisely, the following theorem is true (by B^l is denoted the σ-algebra of Borel sets in R^l).

Theorem. *Whatever standard process*

$$X = (x_t, \xi, \mathcal{M}_t, P_x),$$

given in the measurable space (R^l, B^l) *and satisfying conditions A and B, there exists an equivalent process X, obtained by means of a random change of time in some subprocess of the Wiener process***.

In the case when $\xi \equiv \infty$ and in condition A the equality sign stands, this theorem was proved by Mackeene and Tanaka ([6]). The supposition of its validity in the general case was expressed by E. B. Dynkin ([4]).

1. Denote by

$$\bar{X} = (\bar{x}_t, \bar{\xi}, \bar{\mathcal{M}}_t, \bar{P}_x)$$

some subprocess of the Wiener process, and let $\bar{\tau}_U$ be the moment of first exit of the process $\bar{X}$ from the domain U, while

$$\bar{\pi}_U(x, \Gamma) = \bar{P}_x\{\bar{x}(\bar{\tau}_U) \in \Gamma\}.$$

Our immediate aim will be the construction of such a process $\bar{X}$ for which the measures $\bar{\pi}_U(x, \Gamma)$ coincide with $\pi_U(x, \Gamma)$ for all domains U and all $x \in R^l$. Everywhere, unless the contrary is stated, it is assumed that $l \geq 2$.

* For the terminology, see ([3],[4]). The class of processes considered here was investigated in ([4]).

** Recall that $\tau_U = \xi$, if $x_t \in U$ for all t $(0 \leq t < \xi)$, and

$$\tau_U = \inf(t : t > 0,\ x_t \notin U)$$

otherwise.

*** Two Markov processes given on one and the same measurable space are called equivalent if their transition functions coincide. A homogeneous, right-continuous, strictly Markov process

$$X = (x_t, \xi, \mathcal{M}_t, P_x),$$

given in (R^l, B^l), is called standard if $\mathcal{M}_t \supset \mathcal{N}_{t+0}$ and if, for any $x \in R^l$ and any nondecreasing sequence of random variables τ_n, not depending on the future, $x(\tau_n)$ as $n \to \infty$ tends to $x(\tau)$, where

$$\tau = \lim_{n\to\infty} \tau_n,$$

P_x-almost surely on the set

$$\Omega_1 = \{\tau < \xi\}.$$

Denote by $\rho(x, y)$ the distance between points x and y in R^l, and consider the sequence $\tau_k^{(n)}$, defined as follows. We set $\tau_0^{(n)} \equiv 0$. If $\tau_k^{(n)}$ has already been defined, then $\tau_{k+1}^{(n)}$ is set equal to the lower bound of the times t such that $t > \tau_k^{(n)}$ and

$$\rho(x(t), x(\tau_k^{(n)})) \geq \sqrt{l/2^{n+1}},$$

provided that the set of such t is nonempty; otherwise $\tau_{k+1}^{(n)} = \zeta$. The random variables $x(\tau_k^{(n)})$, for any fixed n, form a Markov chain.

Next, put $x_n(t) = x(\tau_k^{(n)})$, if $\tau_k^{(n)} < \zeta$ and $k\cdot 2^{-n} \le t < (k+1)2^{-n}$ (when $\tau_k^{(n)} \ge \zeta$ and $t \ge k \cdot 2^{-n}$, the quantity $x_n(t)$ is not defined). The random functions $x_n(t)$ are trajectories of a certain nonhomogeneous Markov process

$$X_n = (x_n(t), \xi_n, \mathcal{M}_t^s(n), P_{s,x}^{(n)}),$$

defined on the same set of elementary events as X, and having transition function

$$P_n(s, x, t, \Gamma) = P_x\{x(\tau_v^{(n)}) \in \Gamma\},$$

where v is the difference of the integer parts of the numbers $2^n t$ and $2^n s$. In an analogous way we construct the chains $\hat{x}(\hat{\tau}_k^{(n)})$ and the process $\hat{X}_n$ with transition function $\hat{P}_n(s, x, t, \Gamma)$, starting from the Wiener process. From consideration of the sequences $x(\tau_k^{(n)})$ it is not difficult to derive that, for any $x \in R^l$, the trajectory $x_t(\omega)$ is continuous in t $(0 \le t < \zeta)$ P_x-almost surely.

The transition function of the desired process $\overline{X}$ will subsequently be constructed as the limit $P_n(0, x, t, \Gamma)$.

It is important to note that

$$\hat{P}_n(0, x, t, \Gamma) \ge P_n(0, x, t, \Gamma),$$

and that the study of the distributions of the quantities $\hat{x}(\hat{\tau}_k^{(n)})$ is in an obvious way reduced to the study of the distributions of normalized sums of independent random vectors ξ_i, each of which is uniformly distributed on the circle of unit radius with center at the origin of the coordinates in R^l. Using the theorem of paper ([1]) and the known estimates for the maximum of sums of independent quantities (see ([2]), Ch. 3, Theorem 2.2), we easily obtain*:

a) whatever $T > 0$ and $\varepsilon > 0$ may be, there exist numbers N and K such that for $n > N$ the inequality

$$\hat{P}_n(s, x, t, \Gamma) \le K\lambda(\Gamma) + \varepsilon$$

holds for all $x \in R^l$ and all numbers $s \ge 0,\ t \ge 0$ for which $t - s > T$;

b) for fixed $t \in \Lambda$ and $n \to \infty$, the function $\hat{P}_n(0, x, t, \Gamma)$ tends to the transition function $\hat{P}(t, x, \Gamma)$ of the Wiener process, uniformly in $x \in R^l$ and $\Gamma \in B^l$;

c) $\delta > 0$ can be chosen so that, for all sufficiently large n, the quantity $\alpha_n^\varepsilon(\delta)/\delta$ does not exceed any preassigned number.

We shall say that the Borel measures μ_n converge weakly as $n \to \infty$ to the Borel measure μ, if

$$\int f\, d\mu_n \to \int f\, d\mu$$

for every continuous function $f(x)$ tending to zero at infinity. Denote by τ_n' the first exit time of X_n from U ($U \in C$), and put

$$\mu_n(x, \Gamma) = P_{0,x}^{(n)}\{x_n(\tau_n') \in \Gamma\}.$$

Lemma 1. Whatever $x \in U$ ($U \in C$) may be, the measures $\mu_n(x, \Gamma)$ converge weakly to $\pi_U(x, \Gamma)$ as $n \to \infty$.

From assertions a) and c), Lemma 1, and estimate (6.28) of book ([3]), one may conclude that

d) for any fixed $x \in R^l$, $t \geq 0$, and $\varepsilon > 0$, one can indicate so small an $s > 0$ that, for all sufficiently large n, the following will be fulfilled—

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* In what follows, λ is the ordinary Lebesgue measure in R^l; $V_\varepsilon(x)$ is the exterior of the ε-neighborhood of the point x; $\alpha_n^\varepsilon(\delta)$ is the upper bound of the values of $\hat{P}_n(s, x, t, V_\varepsilon(x))$ for $x \in R^l$ and $t - s \leq \delta$; C is the collection of domains whose boundaries are $(l-1)$-dimensional smooth manifolds of class 2; Λ is the collection of nonnegative dyadic-rational numbers.

the inequalities hold

$$P_n(0, x, s, V_\varepsilon(x)) < \varepsilon, \qquad P_{0,x}^{(n)}\{\xi_n > s\} > 1 - \varepsilon, \qquad P_{0,x}^{(n)}\{t + \dot{s} > \xi_n > t\} < \varepsilon. \tag{1}$$

2. Let us proceed to the construction of the transition function of the desired process $\overline{X}$. Consider the family of functions

$$\Phi_n(s, x, t, \Gamma) = \int \widehat{P}(s, x, dy)\, P_n(s, y, t, \Gamma) \qquad (n \geq 0,\ t \geq s,\ \Gamma \in B^l).$$

For any fixed $s > 0$ this family of functions is equicontinuous in x. Moreover, if $\varepsilon > 0$, $s_0 > 0$, and $x \in R^l$ are fixed, then one can choose $s > 0$ such that, for all $\Gamma \in B^l$, $t > s_0$, and all sufficiently large n,

$$|P_n(0, x, t, \Gamma) - \Phi_n(s, x, t, \Gamma)| < 2\varepsilon. \tag{2}$$

Indeed, assuming s to satisfy the second of inequalities (1) and recalling the Chapman-Kolmogorov equality for X_n, we obtain, for large values of n, that

$$\left| P_n(0, x, t, \Gamma) - \int \widehat{P}_n(0, x, s, dy)\, P_n(s, y, t, \Gamma) \right| < \varepsilon,$$

whence, according to b), our assertion follows.

Lemma 2. There exists a numerical sequence $\{n_k\}$ such that, for any $x \in R^l$ and $t \in \Lambda$, the measures $P_{n_k}(0, x, t, \Gamma)$ converge weakly as $k \to \infty$ to some measure $\overline{P}(t, x, \Gamma)$.

Lemma 2 is easily derived from the assertion: for any $s_0 > 0$ there exists a sequence $\{n_k\}$ such that, for all $x \in R^l$ and all $t > s_0$ $(t \in \Lambda)$, the measures $P_{n_k}(0, x, t, \Gamma)$ converge weakly to some measure $Q(s_0, t, x, \Gamma)$. To prove this assertion, fix $s_0 > 0$, $t > s_0$ $(t \in \Lambda)$, and $n \geq 0$.

For each $x \in R^l$ choose $s = s(n, x)$ $(0 < s(n, x) \leq 2^{-n})$ so as to satisfy (2) with $\varepsilon = 2^{-n-1}$. Then about each x describe an open ball $V(x, n)$ such that, for all points y in it,

$$|\Phi_m(s(n, x), x, t, \Gamma) - \Phi_m(s(n, x), y, t, \Gamma)| < 2^{-n}$$

for any $m \geq 0$, $\Gamma \in B^l$. Keeping n fixed for the time being, from the system of balls $V(x, n)$ extract a countable covering of the whole of R^l. Denote the centers of the balls entering this covering by $x_{n,r}$, and choose $\{n_k\}$ so that our assertion is true for all $x_{n,r}$ for any n and r. The sequence $\{n_k\}$ is the desired one. Indeed, if $x_0 \in R^l$, $2^{-n_0} \leq s(n, x_0)$, and $x_0 \in V(x_{n_0,r_0}, n_0)$, then

$$\left| P_m(0, x_0, t, \Gamma) - P_m(0, x_{n_0,r_0}, t, \Gamma) \right| < 4 \cdot 2^{-n}$$

for all sufficiently large m.

Lemma 2 and assertion a) allow us, from the Chapman-Kolmogorov equality for the process X_n, to conclude that, for all $x \in R^l$ and $\Gamma \in C$ (and consequently also for all $\Gamma \in B^l$),

$$\overline{P}(s + t, x, \Gamma) = \int \overline{P}(s, x, dy)\, \overline{P}(t, y, \Gamma), \tag{3}$$

where $s, t \in \Lambda$. For $t \notin \Lambda$ $(t > 0)$, define the measure $\overline{P}(t, x, \Gamma)$ as the weak limit of the measures $\overline{P}(u, x, \Gamma)$ as $u \downarrow t$ (the correctness of this definition follows from c)). Taking a) into account, one can verify that the measures $\overline{P}(t, x, \Gamma)$ satisfy (3) for all $s, t \geq 0$, and, thus, $\overline{P}(t, x, \Gamma)$ is a transition function. In view of the fact that $\overline{P}(t, x, \Gamma)$ is majorized by the transition function of the Wiener process, the transition function $\overline{P}(t, x, \Gamma)$ corresponds to some subprocess of the Wiener process (7). We shall take this subprocess as $\overline{X}$.

3. It is not difficult to verify that the finite-dimensional distributions of the processes X_{n_k} converge weakly, as $k \to \infty$, to the finite-dimensional distributions of the process $\overline{X}$.

By means of Theorem 3.1.2 from ([5]) we establish that the measures $\mu_n(x, \Gamma)$, for every $U \in C$, converge weakly to $\overline{\pi}_U(x, \Gamma)$, and, consequently, in accordance with Lemma 1, $\pi_U(x,\Gamma) = \overline{\pi}_U(x,\Gamma)$ if $U \in C$. This equality is extended to all domains U with the aid of Lemma 3.

Lemma 3. *Let $Y = (y_t, \xi', \mathcal{M}'_t, Q_x)$ be a standard process satisfying conditions A and B. Let ξ_n be a nondecreasing sequence of random variables independent of the future, and let $\xi = \lim_{n\to\infty} \xi_n$.*

Let the event H consist in the fact that $\sup_{0\le t<\xi} |y_t| < \infty$, where $|y_t|$ denotes the distance of y_t from the origin in R^l. Then the measures

$$\mu_x^{(n)}(\Gamma) = Q_x\{H,\ y(\xi_n) \in \Gamma\}$$

converge weakly to the measures

$$\mu_x(\Gamma) = Q_x\{H,\ y(\xi) \in \Gamma\}$$

as $n \to \infty$.

Now, from a theorem of Blumenthal, Getoor, and McKean one may conclude that the process X is equivalent to some process obtained from $\overline{X}$ by a random change of time. In the case $l = 1$, however, our theorem follows from the works ([8],[9]).

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Note: Figure translations are in progress. See original paper for figures.