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Abstract

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ON THE BEHAVIOR OF THE TRAJECTORIES OF A SYSTEM CLOSE TO AN AUTONOMOUS HAMILTONIAN SYSTEM

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In the present note we shall consider the behavior of the trajectories of the system

$$\dot{x} = \partial H/\partial y + \varepsilon f(x, y, t, \varepsilon), \qquad \dot{y} = -\partial H/\partial x + \varepsilon g(x, y, t, \varepsilon), \tag{1_ε}$$

with respect to which we shall assume that the following conditions are satisfied:

1. $H = H(x, y)$ is an analytic function of x and y in some neighborhood of the origin.
2. $f(x, y, t, \varepsilon)$ and $g(x, y, t, \varepsilon)$ are functions analytic in x, y, and ε in some neighborhood of the point $x = y = \varepsilon = 0$, continuous in t together with the first derivative with respect to t, and periodic in t with period 2π.
3. For $x = y = 0$, $\partial H/\partial x = \partial H/\partial y = 0$,

   $$\Delta = (\partial^2 H/\partial x^2)(\partial^2 H/\partial y^2) - (\partial^2 H/\partial x \partial y)^2 > 0.$$

   The last condition means that for the system (1_0) (the system (1_0) is obtained from (1_ε) for $\varepsilon = 0$) the origin is an equilibrium position of center type, i.e., all trajectories of the system (1_0) beginning in a sufficiently small neighborhood of the origin will be closed.

Let us pass in the system (1_ε) from the variables (x, y) to the variables (H, φ), where $H = H(x, y)$ is the Hamiltonian of the system (1_0), and the function $\varphi = \varphi(x, y)$ is defined in the following way. Let G_0 be the maximal simply connected neighborhood of the origin, entirely filled by closed trajectories of the system (1_0) and containing no other equilibrium positions of this system except the origin. Let, further, (x_0, y_0) be an arbitrary point of the domain G_0, distinct from the origin. Define the curve $L = \{x = \alpha(s),\ y = \beta(s)\}$ as a solution of the system

$$d\alpha/ds = \partial H/\partial \alpha, \qquad d\beta/ds = \partial H/\partial \beta$$

with the following initial conditions: $\alpha(0) = x_0,\ \beta(0) = y_0$. It follows from this definition that the angle of intersection of the curve L with any trajectory of the system (1_0) is equal to $\pi/2$. Hence, and from the definition of the domain G_0, it follows that there exists an interval (s_1, s_2) of variation of s such that, as s varies in this interval, the curve L intersects every trajectory of the system (1_0) lying in the domain G_0, and moreover only once. Finally, let $(x(t,s), y(t,s))$ be a solution of the system (1_0) satisfying the conditions: $x(0,s) = \alpha(s),\ y(0,s) = \beta(s)$ $(s_1 < s < s_2)$. Put $\varphi(x,y)$ equal to the interval of time during which the solution $(x(t,s), y(t,s))$, passing through the point (x,y), having started at $t = 0$ from the corresponding point on the curve L, reaches the point (x,y). It is not difficult to see that in this way $\varphi(x,y)$ can be defined only up to an integer multiple of the period of the solution $(x(t,s), y(t,s))$. However, as is readily verified, each branch of the function $\varphi(x,y)$ will be an analytic function of x and y in a neighborhood of any point of the domain G_0 distinct from the origin. Considering the increment of the function $\varphi(x,y)$ under displacement of the point (x,y) along a trajectory of the system (1_0) during the time Δt, it is not difficult to be convinced that

$\varphi(x,y)$ satisfies the equation

$$\frac{\partial H}{\partial y}\frac{\partial \varphi}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial \varphi}{\partial y} = 1.$$

Hence, taking (1_ε) into account, we obtain

$$\begin{aligned} \dot{H} &= \varepsilon\left(H'_x f + H'_y g\right) = \varepsilon F(H, \varphi, t, \varepsilon), \\ \dot{\varphi} &= 1 + \varepsilon\left(\varphi'_x f + \varphi'_y g\right) = 1 + \varepsilon G(H, \varphi, t, \varepsilon), \end{aligned} \tag{2_ε}$$

where $F(H, \varphi, t, \varepsilon)$ and $G(H, \varphi, t, \varepsilon)$ are analytic functions of H, φ, and ε, except at the point $H = H_0 = H(0,0)$, continuous in t together with the first derivative with respect to t, and periodic in t with period 2π; in the variable φ these functions will also be periodic with period $T(H)$, depending on H.

Let now $H_{m,n} \neq H_0$ be such that $T(H_{m,n}) = 2\pi m/n$, $T'(H_{m,n}) \neq 0$ (m and n are relatively prime integers). In system (2_ε) make the substitution:

$$H = H_{m,n} + \mu h, \quad \varphi = \frac{nt}{2m\pi} T(H) + \psi, \quad \text{where } \mu = \sqrt{\varepsilon}$$

(here and in what follows, for definiteness, we shall assume that $\varepsilon > 0$). After the substitution we obtain

$$dh/dt = \mu A(\psi, t) + \mu^2 P(h, \psi, t, \mu),$$

$$d\psi/dt = \mu a h + \mu^2 Q(h, \psi, t, \mu),$$

where

$$a = -\frac{n}{2m\pi} T'(H_{m,n}) \neq 0,$$

and $A(\psi, t)$, $P(h, \psi, t, \mu)$, and $Q(h, \psi, t, \mu)$ are analytic functions in the variables h, ψ, and μ, continuous in t together with the first derivative with respect to t, and periodic in t with period $2m\pi$. Finally, after the substitution

$$h = u - i\mu m \sum_{p \neq 0} \frac{1}{p} A_p(v) e^{ipt/m}, \quad \psi = v, \quad \text{where} \quad A_p(v) = \frac{1}{2m\pi} \int_0^{2m\pi} A(v, t) e^{-ipt/m}\, dt,$$

we obtain the system

$$\dot{u} = \mu A_0(v) + \mu^2 R(u, v, t, \mu), \quad \dot{v} = \mu a u + \mu^2 S(u, v, t, \mu), \tag{3_μ}$$

where $R(u, v, t, \mu)$ and $S(u, v, t, \mu)$ are analytic functions of u, v, and μ, continuous in t together with the first derivative with respect to t, and periodic in t with period $2m\pi$.

Let us make one essential remark about the function $A_0(v)$. By definition,

$$A_0(v) = \frac{1}{2m\pi} \int_0^{2m\pi} \{-f(x_0(t+v), y_0(t+v), t, 0)\dot{y}_0(t+v)$$

$$+ g(x_0(t+v), y_0(t+v), t, 0)\dot{x}_0(t+v)\}\, dt,$$

where $(x_0(t), y_0(t))$ is a solution of system (1_0), periodic with period $2\pi m/n$, satisfying the condition $H(x_0(t), y_0(t)) \equiv H_{m,n}$. It follows that $A_0(v)$ is an analytic function of v, periodic in v with period $2\pi m/n$. On the other hand, replacing under the integral sign, with the help of which $A_0(v)$ is defined, $t + v$ by t, we obtain that $A_0(v)$ has period 2π. Hence it follows easily that $A_0(v)$ has period $2\pi/n$, i.e., on the interval $[0, 2\pi m/n]$ exactly m periods of the function $A_0(v)$ are contained. In what follows we shall assume of the function $A_0(v)$ that it has only simple zeros and, in particular, is not identically equal to zero. The case when $A_0(v)$ does not vanish at all is elementary. In this case the following theorem holds, characterizing the behavior of the trajectories of system (1_ε) in a certain neighborhood of the curve $H(x, y) = H_{m,n}$.

Theorem 1. Let the function $A_0(v)$, occurring in the system (3_μ), not vanish. Then there exist constants $\mu_0 > 0$, $\delta_0 > 0$, $c_0 > 0$ such that, for any positive $\varepsilon < \mu_0^2$, every solution of the system (1_ε) which at $t = 0$ starts from an arbitrary point of the domain

$$H_{m,n} - \mu\delta_0 < H(x, y) < H_{m,n} + \mu\delta_0,$$

for all $t > c_0/\mu$ will lie in the domain

$$H(x, y) < H_{m,n} - \mu\delta_0,$$

if

$$T'(H_{m,n}) \operatorname{sign} A_0(v) > 0,$$

and in the domain

$$H(x, y) > H_{m,n} + \mu\delta_0,$$

if

$$T'(H_{m,n}) \operatorname{sign} A_0(v) < 0.$$

To investigate the case where the function $A_0(v)$ has simple zeros, we shall try to find an analogy between the system (3_μ) and the system considered in note [1]. To this end, in the system (3_μ) we replace the independent variable t by τ/μ. After the substitution we obtain the system

$$du/d\tau = A_0(v) + \mu R(u, v, \tau/\mu, \mu), \qquad dv/d\tau = au + \mu S(u, v, \tau/\mu, \mu), \quad (4_\mu)$$

which is similar to the system considered in note [1]; however, the presence, in the arguments of the functions $R(u, v, \tau/\mu, \mu)$ and $S(u, v, \tau/\mu, \mu)$, of the singular factor $1/\mu$ does not allow the results of note [1] to be applied directly to the system (4_μ). Nevertheless, we shall try to obtain for the system (4_μ) results analogous to the results of note [1], and with their help to investigate the disposition of the trajectories of the system (1_ε). For this purpose, instead of the system (4_μ), we consider the system

$$\dot{u} = A_0(v) + \mu R(u, v, \omega t, \mu), \qquad \dot{v} = au + \mu S(u, v, \omega t, \mu), \qquad (5_\mu)$$

in which the frequency ω does not depend on μ. However, we shall greatly shorten the path leading to the solution of the problem if, before passing from the system (3_μ) to the system (5_μ), we subject the system (3_μ) to two transformations.

The meaning of the first transformation is contained in the following lemma.

Lemma 1. Let $v = v_r$ be a simple root of the equation $A_0(v) = 0$. Then there exist two functions $p_r(t, \mu)$ and $q_r(t, \mu)$, analytic in a certain disk $|\mu| < \mu_0$, continuous in t together with their partial derivatives with respect to t up to and including the second order, and periodic in t with period $2m\pi$, such that, after making in the system (3_μ) the substitution

$$u \to u + \mu p_r(t, \mu), \qquad v \to v + \mu q_r(t, \mu),$$

we obtain a system of the same form as the system (3_μ), with the additional property that, for $u = 0$, $v = v_r$, the right-hand sides of the resulting system vanish identically.

The second transformation is given by the following formulas:

$$u = \xi - i\mu^2 m \sum_{p\neq 0} \frac{1}{p} R_p(\xi,\eta) e^{ipt/m}, \qquad v = \eta - i\mu^2 m \sum_{p\neq 0} \frac{1}{p} S_p(\xi,\eta) e^{ipt/m},$$

where

$$R_p(\xi,\eta) = \frac{1}{2m\pi} \int_0^{2m\pi} R(\xi,\eta,t,0) e^{-ipt/m}\,dt,$$

$$S_p(\xi,\eta) = \frac{1}{2m\pi} \int_0^{2m\pi} S(\xi,\eta,t,0) e^{-ipt/m}\,dt.$$

After carrying out the transformations described, the system (3_μ) will take the form

$$\dot\xi = \mu A_0(\eta) + \mu^2 R_0(\xi,\eta) + \mu^3 \widetilde{R}(\xi,\eta,t,\mu),$$

$$\dot\eta = \mu a\xi + \mu^2 S_0(\xi,\eta) + \mu^3 \widetilde{S}(\xi,\eta,t,\mu),$$

where

$$A_0(v_r) = R_0(0,v_r) = S_0(0,v_r) = 0, \qquad \widetilde{R}(0,v_r,t,\mu) = \widetilde{S}(0,v_r,t,\mu) = 0.$$

Thus, we have arrived at the system

$$\dot\xi = A_0(\eta) + \mu R_0(\xi,\eta) + \mu^2 \widetilde{R}(\xi,\eta,\omega t,\mu),$$

$$\dot\eta = a\xi + \mu S_0(\xi,\eta) + \mu^2 \widetilde{S}(\xi,\eta,\omega t,\mu), \tag{6_μ}$$

where the functions $\widetilde{R}(\xi,\eta,\omega t,\mu)$ and $\widetilde{S}(\xi,\eta,\omega t,\mu)$ are periodic in $\theta = \omega t$ with period $2m\pi$. For this system the following two theorems are valid.

Theorem 2. Let $\eta = v_r$ be a root of the equation $A_0(\eta) = 0$ such that $aA_0'(v_r) > 0$. Then there exist $\mu_0 > 0$, $\delta_0 > 0$, $c_0 > 0$ such that, for any complex μ, ξ_0 satisfying the inequalities $|\mu| < \mu_0$, $|\xi_0| < \delta_0$, for any real ω satisfying the inequality $-c_0/|\mu|^2 < \omega < c_0/|\mu|^2$, and for any t_0, there exists a unique solution $(\xi_\mu(t), \eta_\mu(t))$ of system (6_μ) such that $\xi_\mu(t_0) = \xi_0$; for all $t \geqslant t_0$

$$\frac{d}{dt}|\xi_\mu(t)| < 0, \qquad \frac{d}{dt}|\eta_\mu(t) - v_r| < 0; \qquad |\xi_\mu(t)| + |\eta_\mu(t) - v_r| \to 0 \quad \text{as } t \to \infty.$$

Theorem 3. Let $\eta = v_r$ be a root of the equation $A_0(\eta) = 0$ such that $aA_0'(v_r) > 0$. Then there exist $\mu_1 > 0$, $\delta_1 > 0$ and $c_1 > 0$ ($\mu_1 \leqslant \mu_0$, $\delta_1 \leqslant \delta_0$, $c_1 \leqslant c_0$) such that, for any complex μ and ξ_0 satisfying the inequalities $|\mu| < \mu_1$, $|\xi_0| < \delta_1$, for any real ω satisfying the inequality $-c_1/|\mu|^2 < \omega < c_1/|\mu|^2$, and for any t_0, the solution $(\xi_\mu(t), \eta_\mu(t))$ of system (6_μ) satisfying the conditions: $\xi_\mu(t_0) = \xi_0$, for all $t \geqslant t_0$

$$\frac{d}{dt}|\xi_\mu(t)| < 0 \quad \text{and} \quad \frac{d}{dt}|\eta_\mu(t) - v_r| < 0, \qquad |\xi_\mu(t)| + |\eta_\mu(t) - v_r| \to 0 \quad \text{as } t \to \infty,$$

possesses continuous partial derivatives with respect to $\sigma = \operatorname{Re}\mu$ and $\nu = \operatorname{Im}\mu$, which satisfy the Cauchy–Riemann conditions for all $t \geqslant t_0$, $\sigma^2 + \nu^2 < \mu_1^2$ and $-c_1/|\mu|^2 < \omega < c_1/|\mu|^2$.

On the basis of Theorem 3 we can find the boundary trajectories of system (6_μ) in the form of series in powers of the parameter μ, which will converge for all $|\mu| < \mu_1$ and $-c_1/|\mu|^2 < \omega < c_1/|\mu|^2$. Setting $\omega = 1/\mu$ ($\mu > 0$) in the obtained series, we obtain series which will converge for all real μ satisfying the inequality $0 < \mu < \min(\mu_1, c_1)$. Analysis of the series obtained makes it possible to reveal a number of interesting regularities concerning the behavior of the trajectories of system (1_ε).

Thus, for example, it can be shown that, under certain conditions, in the domain G_0 there will exist families consisting of a finite number of pairwise nonintersecting domains G_k ($k = 1, \dots, m$) such that any solution of system (1_ε) that leaves the domain G_m at $t = t_0$ enters the domain G_k at $t = t_0 + 2k\pi$ ($0 < k \leqslant m$), and, for sufficiently small $\varepsilon \neq 0$, such families are possible with an arbitrarily large number m.

In conclusion, we note that the series obtained for the boundary trajectories of system (6_μ) for $\omega = 1/\mu$ have a rather complicated structure. In these series there will necessarily be terms of the form $e^{-\alpha/\mu}$ ($\alpha > 0,\ \mu > 0$). This, in particular, means that the widely known method of averaging[2] cannot be adapted to the determination of the boundary trajectories of system (6_μ) and, consequently, to the analysis of system (1_ε).

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2. N. N. Bogolyubov, Yu. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, 1958.

Note: Figure translations are in progress. See original paper for figures.