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Abstract

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ON A DYNAMICAL SYSTEM CONNECTED WITH THE DISTRIBUTION OF THE FRACTIONAL PARTS OF A POLYNOMIAL OF THE SECOND DEGREE

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A dynamical system connected with the distribution of the fractional parts of a linear function was considered by A. Ya. Khinchin. In ([1]), the ergodicity of the dynamical system (Ω, T^x, μ) is established by elementary methods, where $\Omega = [0, 1)$; $T^x(\alpha) = \{\alpha + x\gamma\}$; $x = 1, 2, ...$; γ is a fixed irrational number; μ is Lebesgue measure on Ω; $\{\beta\}$ is the fractional part of the number β.

We consider a dynamical system connected with the distribution of the fractional parts of a polynomial of the second degree. Let Ω be the unit square $0 \leq \alpha < 1,\ 0 \leq \beta < 1$; μ be Lebesgue measure on Ω. Define a mapping T of Ω onto itself by the formula

$$T(\alpha, \beta) = (\{\alpha + 2\beta + \gamma\}, \{\beta + \gamma\}),$$

where γ is a fixed irrational number. It is easily established that for any natural x and y

$$T^x(\alpha, \beta) = (\{\alpha + 2\beta x + \gamma x^2\}, \{\beta + \gamma x\}),$$

$$T^x T^y(\alpha, \beta) = T^{x+y}(\alpha, \beta).$$

The transformation T is one-to-one, and μ is an invariant measure with respect to T. It is not difficult to write down formulas for the transformation T' of the dynamical system $(\Omega', {T'}^x, \mu')$; $x = 1, 2, ...$, Ω' is the unit hypercube $0 \leq \alpha_i < 1$; $i = 1, 2, \dots, n$; μ' is n-dimensional Lebesgue measure, connected with the distribution of the fractional parts of a polynomial of the n-th degree.

Theorem 1. *Let s be a fixed natural number. The dynamical system (Ω, T^{sx}, μ), $x = 1, 2, ...,$ is ergodic.*

Proof. Denote by U the operator in the space $L^2_{\Omega,\mu}$ of complex-valued functions integrable over the square with respect to the measure μ:

$$Uf(\alpha,\beta) = f(T^s(\alpha,\beta)).$$

The operator U is unitary. It is enough to prove that for any function $f(\alpha,\beta)$ from $L^2_{\Omega,\mu}$, from the equality

$$Uf(\alpha,\beta) = f(\alpha,\beta)$$

for almost all points (α,β) it follows that for almost all (α,β)

$$f(\alpha,\beta) = C,$$

where $C = \text{const}$. Let

$$f(\alpha,\beta) \sim \sum_{m_1,m_2=-\infty}^{\infty} c_{m_1,m_2} e^{2\pi i(m_1\alpha+m_2\beta)}$$

the expansion of $f(\alpha,\beta)$ in a Fourier series. Making a change of variables in the integrals

$$c_{m_1,m_2} = \iint_\Omega f(\alpha,\beta) e^{-2\pi i(m_1\alpha+m_2\beta)}\, d\mu,$$

we obtain, by virtue of the invariance of the measure μ, the relations

$$c_{m_1,m_2} = e^{-2\pi i\gamma(m_1 s^2+m_2 s)} c_{m_1,\,2sm_1+m_2}.$$

Suppose there exists a pair of integers m_1, m_2 with $m_1 = 0,\ m_2 \neq 0,\ c_{m_1,m_2} \neq 0$ –this is impossible by the irrationality of γ. The case in which in the expansion of $f(\alpha,\beta)$ there exists a nonzero c_{m_1,m_2} with $m_1 \neq 0$ contradicts Bessel' s inequality for $f(\alpha,\beta)$, since then this expansion contains an infinite number of nonzero coefficients, equal to one another in modulus, of the form $c_{m_1,\,2ksm_1+m_2}$, $k = 1, 2, \ldots$. Consequently, $f(\alpha,\beta) = c_{0,0}$ for almost all (α,β).

Corollary. *Let $f(\alpha,\beta) \in L^1_{\Omega,\mu}$. Then for any natural number s*

$$\lim_{p\to\infty} \frac{1}{p} \sum_{x=0}^{p-1} f\big(\{a+2\beta sx+s^2x^2\gamma\}, \{\beta+sx\gamma\}\big) = \iint_\Omega f(\alpha,\beta)\, d\mu.$$

The proof follows from the Birkhoff–Khinchin theorem (([2]), p. 31). Theorem 2 gives information on the question of the existence of a singly mixing transformation that is not doubly mixing (see ([2]), p. 133).

Theorem 2. *Let M be the set of functions from $L^2_{\Omega,\mu}$ that depend only on α:*

$$\varphi(\alpha,\beta)=\varphi(\alpha),$$

with mean value equal to zero. The transformation T is singly mixing on M and is not doubly mixing.

Proof. Let $\varphi_1(\alpha,\beta),\varphi_2(\alpha,\beta)\in M$

$$\varphi_1(\alpha,\beta)\sim\sum_{m_1=-\infty}^{\infty} c_{m_1}e^{2\pi i m_1\alpha},\qquad \varphi_2(\alpha,\beta)\sim\sum_{m_2=-\infty}^{\infty}{}' d_{m_2}e^{2\pi i m_2\alpha}$$

$$\left(\sum_{m=-\infty}^{\infty}{}'\text{ denotes the absence in the sum of the term with zero index}\right)$$

–be their Fourier series, and let $l>0$ be an integer. We have ($s=1$)

$$\iint_\Omega U^l\varphi_1(\alpha,\beta)\varphi_2(\alpha,\beta)\,d\mu=$$

$$=\sum_{m_1=-\infty}^{\infty}{}'\sum_{m_2=-\infty}^{\infty}{}' c_{m_1}d_{m_2}e^{2\pi i m_1 l^2\gamma}\iint_\Omega e^{2\pi i[(m_1+m_2)\alpha+2m_1 l\beta]}\,d\mu=0.$$

Thus, for all $l\geq 1$,

$$\iint_\Omega U^l\varphi_1(\alpha,\beta)\varphi_2(\alpha,\beta)\,d\mu=\iint_\Omega\varphi_1(\alpha,\beta)\,d\mu\iint_\Omega\varphi_2(\alpha,\beta)\,d\mu.$$

This means precisely that T is mixing on M.

Next, let $\varphi_0(\alpha,\beta)=e^{2\pi i\alpha}$, $\varphi_1(\alpha,\beta)=e^{2\pi i\alpha}$, $\varphi_2(\alpha,\beta)=e^{-4\pi i\alpha}$. Take the sequence of triples of natural numbers

$$(k_l^{(0)},k_l^{(1)},k_l^{(2)})=(2l,4l,3l),\qquad l=1,2,\ldots$$

The conditions $\varphi_i(\alpha,\beta)\in M,\ i=1,2,3,$

$$\lim_{l\to\infty}\min_{i<j}\left|k_l^{(i)}-k_l^{(j)}\right|=\infty$$

are satisfied,

$$\iint_\Omega U^{k_l^{(0)}} \varphi_0(\alpha,\beta)\, U^{k_l^{(1)}} \varphi_1(\alpha,\beta)\, U^{k_l^{(2)}} \varphi_2(\alpha,\beta)\, d\mu = e^{2\pi i l^2/2^\gamma},$$

$$\lim_{l\to\infty} e^{2\pi i l^2/2^\gamma}$$

does not exist; on the other hand,

$$\iint_\Omega \varphi_0(\alpha,\beta)\, d\mu \cdot \iint_\Omega \varphi_1(\alpha,\beta)\, d\mu \cdot \iint_\Omega \varphi_2(\alpha,\beta)\, d\mu = 0.$$

Consequently, T is not doubly mixing on M.

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References Cited

[1] A. Ya. Khinchin, *Matem. sborn.*, **41**, 11 (1934).
[2] P. Halmos, *Lectures on Ergodic Theory*, Moscow, 1959.

Note: Figure translations are in progress. See original paper for figures.