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Abstract

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ON QUASIANALYTIC CLASSES OF FUNCTIONS OF SEVERAL VARIABLES

(Presented by Academician S. N. Bernstein on 17 IV 1962)

This article adjoins the joint work of V. I. Matsaev and the author (^{1}), in which some generalizations of the well-known Carleman theorem on quasianalytic classes were obtained for the case of functions of any finite number of independent variables.

1. Following (^{1}), we shall call a certain class of functions quasianalytic I if it contains no function, different from identically zero, that vanishes at some point together with all its derivatives. We shall call a class of functions quasianalytic II if it contains no nontrivial finite function, i.e., a function different from identically zero and equal to zero everywhere outside some finite domain.

Let D be a certain domain in the plane R_2 of the real variables x, y*. Further, let $\{m_{p,q}\}$ be a certain double sequence of nonnegative numbers. Denote by $C_D(m_{p,q})$ the totality of all functions infinitely differentiable in the domain D for which

$$\sup_{(x,y)\in D} \left| \frac{\partial^{p+q} f(x,y)}{\partial x^p \partial y^q} \right| < M r^p s^q m_{p,q},$$

where the numbers M, r, s are specific to each function of the class $C_D(m_{p,q})$. This definition is analogous to the definition of the class $C(m_n)$ of functions of one variable. Let us note that, as was shown by S. Mandelbrojt (^{2}), for the classes $C(m_n)$, quasianalyticity I and II are equivalent. In the case of functions of several variables there is no such equivalence. This was shown in (^{1}) by comparing the conditions for quasianalyticity II of the class $C_D(m_{p,q})$, obtained earlier by P. Lelong (^{3}), and the conditions for quasianalyticity I of the class $C_D(m_{p,q})$, obtained in (^{1}).

2. In the case when $D = R_2$, the following was obtained in (^{1}).

Theorem 1. *For quasianalyticity I of the class $C_D(m_{p,q})$ it is necessary and sufficient that each of the sequences $\{m_{p,0}\}$, $\{m_{0,q}\}$ generate a quasianalytic class of functions of one variable.*

We shall prove here this theorem for the case of an arbitrary domain D. For this we shall need the following lemmas:

Lemma 1. *Let the function $f(x)$ be infinitely differentiable on the interval $[-1,1]$. Put $\psi(x)=f(\cos x)$. Then, if*

$$\sup_{-1\le x\le 1}\left|f^{(p)}(x)\right|<M_p,\qquad p=0,1,2,\dots,$$

then

$$\sup_{-\infty<x<\infty}\left|\psi^{(p-1)}(x)\right|<c^pM_p+4M_0p^p,$$

where c is a constant independent of the function.

* Consideration of a larger number of variables differs from that set out in §§ 1 and 2 only by a complication of notation.

For the proof of the lemma we use a device which was applied by S. N. Bernstein (^{4}) in proving his theorem on the influence of the rate of approximation of a function by polynomials on its differential properties.

Denote by $B_n(x)$ the trigonometric polynomial of degree n which deviates least from the function $\psi(x)$, and by $s_n(x)$ the partial sum of the Fourier series of the function $\psi(x)$. Also put

$$u_k(x)=B_{2^{k+1}p}(x)-B_{2^kp}(x).$$

Then

$$\psi(x)=s_p(x)+(B_{2p}(x)-s_p(x))+\sum_{k=1}^{\infty}u_k(x).$$

Let us estimate the terms of the series in modulus. For $k\geqslant 1$ and any $x\in(-\infty,\infty)$ we have

$$|u_k(x)|\leqslant|B_{2^{k+1}p}(x)-\psi(x)|+|B_{2^kp}(x)-\psi(x)|\leqslant 2E^T_{2^kp},$$

where

$$E^T_n=\max_{-\infty<x<\infty}|\psi(x)-B_n(x)|.$$

It is known (see, for example, (^{5}), p. 157) that $E^T_n=E_n$, where E_n is the best approximation of the function $f(x)$ by algebraic polynomials of degree not exceeding n. On the other hand, by Jackson' s theorem (see, for example, (^{5})), beginning with $n\geqslant p$, we have

$$E_n<c_1^pM_pn^{-p},$$

where the constant $c_1 < 24e$. Consequently,

$$\max_{-\infty<x<\infty} |u_k(x)| \leqslant 2c_1^p p^{-p} 2^{-kp} M_p.$$

From the same considerations, and also by virtue of Lebesgue' s theorem (see, for example, (5,6)) on the approximation of a periodic function by partial sums of its Fourier series, we obtain

$$\max_{-\infty<x<\infty} |B_{2p}(x) - s_p(x)| \leqslant 2 \max_{-\infty<x<\infty} |\psi(x) - s_p(x)| \leqslant$$

$$\leqslant 2(3 + \ln p) E_p^T \leqslant 2(3 + \ln p) c_1^p p^{-p} M_p.$$

Since each coefficient of the Fourier series of the function $\psi(x)$, obviously, does not exceed $2M_0$, it follows that $|s_p(x)| \leqslant 4pM_0$ for any x.

We now estimate $\psi^{(p-1)}(x)$. By S. N. Bernstein' s inequality for trigonometric polynomials, for any $x \in (-\infty, \infty)$ we have

$$|\psi^{(p-1)}(x)| \leqslant |s_p^{(p-1)}(x)| + \left|(B_{2p}(x) - s_p(x))^{(p-1)}\right| + \sum_{k=1}^{\infty} |u_k^{(p-1)}(x)| \leqslant$$

$$\leqslant 4p^p M_0 + 2^{p+1} c_1^p M_p + \sum_{k=1}^{\infty} (p2^{k+1})^{p-1} p^{-p} 2^{-pk} M_p \leqslant 4p^p M_0 + c^p M_p.$$

The lemma is proved.

Lemma 2. If the class $C(m_n)$ is quasianalytic, then the class $C(m_n + n^n)$ is also quasianalytic *.

Proof. Let $C(m_n)$ be a quasianalytic class. Two cases are possible: either $m_n \geqslant n^n$ for all n, starting from some n, and then the assertion of the lemma is obvious; or there exists such an infinite a sequence of indices n_k, such that $m_{n_k} < n_k^{n_k}$ for every k. Note that in this case one may, without loss of generality, assume that $n_{k+1} > 2n_k$ for every k. Put

$$\beta_n = \inf_{k \geq n} \sqrt[k]{m_k + k^k}.$$

By Carleman' s theorem (see, for example, (8), p. 104) on quasianalytic classes of functions, in order to prove the lemma it is enough to show that

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* It is known (7) that the sum of two functions from different quasianalytic classes may turn out to be a function belonging to no quasianalytic class. At the same time, by Lemma 2, it follows that the sum of a function from a quasianalytic class and an analytic function is a function from a quasianalytic class.

$$\sum_{n=1}^{\infty} \frac{1}{\beta_n} = \infty.$$

Observe that $\beta_{n_k} \le n_k \sqrt[n_k]{2} < 2n_k$, and that $\beta_{n_1} \ge \beta_{n_2}$ for $n_1 \le n_2$. Consequently,

$$\sum_{n=1}^{\infty} \frac{1}{\beta_n} = \sum_{k=0}^{\infty} \sum_{i=n_k+1}^{n_{k+1}} \frac{1}{\beta_i} \ge \sum_{k=0}^{\infty} \frac{n_{k+1} - n_k}{n_{k+1}} = \infty.$$

The lemma is proved.

Now let $f(x, y) \in C_D(m_{p,q})$ vanish, together with all its derivatives, at the point (x_0, y_0), and let the sequences $\{m_{p,0}\}$ and $\{m_{0,q}\}$ generate quasianalytic classes of functions of one variable. Without loss of generality one may assume that D is the square $\{-1 \le x, y \le 1\}$ and that $x_0 = 0$, $y_0 = 0$. Consider the function $\psi(x, y) = f(\cos x, \cos y)$. It is easy to see that this function belongs to some class $C_{R_2}(M_{p,q})$, for which, by Lemma 1, $M_{p-1,0} \le 4p^p m_{0,0} + c^p m_{p,0}$ and $M_{0,q-1} \le 4q^q m_{0,0} + c^q m_{0,q}$. By Lemma 2 the sequences $\{M_{p,0}\}$, $\{M_{0,q}\}$ generate quasianalytic classes of functions of one variable.* Since the function $\psi(x, y)$, together with all its derivatives, vanishes at the point $(\pi/2, \pi/2)$, it follows from Theorem 1, proved in (1) for the case $D = R_2$, that $\psi(x, y) \equiv 0$. Hence $f(x, y) \equiv 0$.

Necessity is verified in exactly the same way as in the case $D = R_2$.

3. P. Lelong (3) showed that for quasianalyticity of the class II $C_{R_2}(m_{p,q})$ it is necessary and sufficient that the sequence

$$l_n = \min_{p+q=n} \{m_{p,q}\}$$

generate a quasianalytic class of functions of one variable. By the definition of a quasianalytic class II, a nontrivial function from it cannot be equal to zero everywhere outside a finite domain. However, it may be equal to zero everywhere inside some domain. It turns out that such domains of zeros cannot be arbitrary; more precisely, the following theorem holds.

Theorem 2. *Let L be some curve, and let D be the smallest rectangle with sides parallel to the coordinate axes that contains L. Suppose further that $f(x, y) \in C_D(m_{p,q})$, where the class $C_D(m_{p,q})$ is quasianalytic II. Then $f(x, y) = 0$ for $(x, y) \in D$.*

The proof of this theorem is based on the following lemmas:

Lemma 3. *If $|f^{(p)}(x)| \le M_p$ for $x \in [0, a]$, $p = 0, 1, 2, \ldots$, and $f^{(p)}(\xi) = 0$, where $\xi \in [0, a]$, $p = 0, 1, 2, \ldots$, then for $x \in [-\sqrt{a}, \sqrt{a}]$*

$$\left|[f(x^2)]^{(p)}\right| \leq (2\sqrt{a}\,e^{\sqrt{2}})^p M_p.$$

Lemma 4. *Suppose the function $f(x, y)$ satisfies the hypotheses of Theorem 2, with $D = \{0 \leq x \leq a,\ 0 \leq y \leq b\}$. Then*

$$f(x^2, y) \in C_{D_1}(m_{p,q}), \qquad f(x, y^2) \in C_{D_2}(m_{p,q}), \qquad f(x^2, y^2) \in C_{D_3}(m_{p,q}),$$

where D_1, D_2, D_3 are respectively the rectangles $\{|x| \leq \sqrt{a},\ 0 \leq y \leq b\}$, $\{0 \leq x \leq a,\ |y| \leq \sqrt{b}\}$, $\{|x| \leq \sqrt{a},\ |y| \leq \sqrt{b}\}$.

The proof of Lemma 3 essentially repeats the proof of one lemma of Mandelbrojt ((2), p. 24).

Lemma 4 is easily proved with the aid of Lemma 3.

Let us outline the proof of the theorem. Suppose that there exists a point (x_1, y_1) at which $f(x_1, y_1) \neq 0$. Then, with the aid of Lemma 4, one constructs a function $\psi(x, y) \in C_{D^*}(m_{p,q})$, where D^* is one of the rectangles D_1, D_2, D_3, which vanishes together with all its derivatives on some closed curve L^* and is nonzero at least at one point inside the domain K bounded by the curve L^*. But then the function equal to $\psi(x, y)$ for $(x, y) \in K$ and equal to zero for $(x, y) \notin K$ also belongs to the class $C_{D^*}(m_{p,q})$ and is a nontrivial finite function, which contradicts the quasi-analyticity II of the class $C_{D^*}(m_{p,q})$. The theorem is proved.

Let us note a simple but, in our opinion, interesting consequence. Let the curve L intersect every straight line parallel to either of the coordinate axes. Then, in order that the class $C_{R_2}(m_{p,q})$ contain no nontrivial function vanishing on the curve L together with all its derivatives, it is necessary and sufficient that the class $C_{R_2}(m_{p,q})$ be quasi-analytic II.

It can also be proved that if the curve L is finite, then, in order that in the class $C_{R_2}(m_{p,q})$ there be no nontrivial function equal to zero together with all its derivatives on L, it is necessary and sufficient that the class $C_{R_2}(m_{p,q})$ be quasi-analytic I.

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* Estimates for the derivatives of the functions $\psi(x) = f(\cos x)$, different from those obtained in Lemma 1, are available in (8). Relying on these estimates, we were not able to show that from the membership of $f(x)$ in a quasianalytic class there follows the membership of $\psi(x)$ in some quasianalytic class.

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Note: Figure translations are in progress. See original paper for figures.

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