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Abstract

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ON THE DOUBLE-LAYER POTENTIAL FOR IRREGULAR DOMAINS

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The most general classes of domains known so far for which the theory of potential has been developed were defined in the classical works of A. M. Lyapunov (see (1)) and T. Carleman (2). For the plane case, J. Radon (3) constructed a theory of potential for a broad class of domains whose boundaries are curves with bounded variation of rotation. We note that there exist Lyapunov curves for which the variation of rotation is not bounded.

Below, Radon’ s results concerning the double-layer potential are extended to a certain class of plane and spatial domains, which includes Lyapunov and Carleman surfaces, and in the plane case also Radon curves.

Let us consider the three-dimensional case. The corresponding arguments for plane curves are only simplified.

Let $\Omega \subset E^3$ be an open domain bounded by a closed surface Γ in E^3. Project Γ from an arbitrary point P onto the unit sphere S_P with center at P. This projection defines a continuous mapping $\pi_P(\Gamma)$ of the surface $\Gamma \setminus P$ into S_P.

We require that the surface Γ satisfy the single condition: the **absolute variation** (4) v_P^0 **of the mapping** $\pi_P(\Gamma)$ **is uniformly bounded for all points** $P \in E^3$, i.e.

$$\sup_{\{F_i\},\, P} \sum_i \mu(\pi_P(F_i)) \leq M = \text{const} < \infty, \tag{1}$$

where μ is area on S_P; $\{F_i\}$ is a finite collection of closed, nonintersecting subsets of Γ.

Condition (1) is satisfied for Radon curves, Lyapunov surfaces, and Carleman surfaces. However, there exist smooth surfaces for which inequality (1) is not satisfied. From a known theorem of area theory (5) it follows easily that a surface Γ satisfying condition (1) has finite Lebesgue area. The finiteness of the area of the surface Γ is used essentially in the proofs.

In order to transfer to the irregular case the concept of the solid angle $\omega(P, \mathcal{E})$ under which a Borel set $\mathcal{E} \subset \Gamma$ is seen from the point P, we use, as applied to $\pi_P(\Gamma)$, the following expressions introduced by A. V. Pogorelov (^{4}) for the positive, negative, and total variation of a continuous mapping through the multiplicity function:

$$v_P^+(\mathcal{E}) = \int_{S_P} n_{\mathcal{E}}^+(Y)\,dY, \qquad v_P^-(\mathcal{E}) = \int_{S_P} n_{\mathcal{E}}^-(Y)\,dY, \qquad v_P(\mathcal{E}) = v_P^+(\mathcal{E}) - v_P^-(\mathcal{E}),$$

where $n_{\mathcal{E}}^+(Y)$, $n_{\mathcal{E}}^-(Y)$ are the numbers of preimages in $\mathcal{E}$ of the point $Y \in S_P$, whose topological index (ind) with respect to $\pi_P(\Gamma)$ is equal to $+1$ and -1, respectively. (Here we assume that S_P and Γ are oriented in the same way.)

Let, by definition, $\omega(P, \mathcal{E}) = v_P(\mathcal{E})$ for $P \in \mathcal{E}$. For $P \in \mathcal{E}$ put

$$\omega(P, \mathcal{E}) = \omega(P, \mathcal{E} \setminus P) + \omega(P, P),$$

where

$$\omega(P, P) = 2\pi - \omega(P, \Gamma \setminus P).$$

We note the following properties of the solid angle $\omega(P, \mathcal{E})$.

1. *The function* $\omega(P, \mathcal{E})$ *is completely additive on the ring of Borel sets.* This property follows directly from the complete additivity of $v_P(\mathcal{E})$, proved by A. V. Pogorelov, and from the definition of $\omega(P, \mathcal{E})$.

2. *The equalities hold*

$$\omega(P, \Gamma) = \begin{cases} 4\pi, & \text{if } P \in \Omega, \\ 2\pi, & \text{if } P \in \Gamma, \\ 0, & \text{if } P \in C\overline{\Omega}. \end{cases}$$

A point $X \in \Gamma$ is called regular with respect to $\pi_P(\Gamma)$ if there exists a neighborhood $\mathcal{E} \subset \Gamma$ of the point X such that the ray PX has no common points with $\mathcal{E} \setminus X$. We divide the regular points $X \in \Gamma$ into three classes Γ^+, Γ^-, Γ^0, depending on whether the ray PX at the point X passes from Ω into $C\overline{\Omega}$, from $C\overline{\Omega}$ into Ω, or remains in Ω or $C\overline{\Omega}$.

The proof of property 2 is easily obtained by using the results of paper (^{4}) and the following simple assertion.

Lemma 1. *Let X be a regular point. If $X \in \Gamma^+$, then* $\operatorname{ind} X \geqslant 1$*; if $X \in \Gamma^-$, then* $\operatorname{ind} X \leqslant -1$. *If* $\operatorname{ind} X = \pm 1$, *then $X \in \Gamma^+$ or $X \in \Gamma^-$, respectively. The equality* $\operatorname{ind} X = 0$ *is necessary and sufficient for $X \in \Gamma^0$.*

3. *The estimate is established*

$$|\omega(P_1, \mathcal{E}) - \omega(P_2, \mathcal{E})| \leqslant \frac{K s(\mathcal{E})}{d^3}\,|P_1 P_2|, \tag{2}$$

where d is the smaller of the distances from P_1 and P_2 to $\mathcal{E}$; K is an absolute constant, and $s(\mathcal{E})$ is the area of the set $\mathcal{E}$.

4. *The inequality holds*

$$\left| \sup_{\substack{\bigcup \mathcal{E}_i = \Gamma \\ \mathcal{E}_i \cap \mathcal{E}_j = \emptyset}} \sum_i |\omega(P, \mathcal{E}_i)| \right| \leqslant v_P^0 + 2\pi.$$

By the double-layer potential with continuous density $f(X)$ we shall mean the function

$$W(P) = \frac{1}{2\pi} \int_\Gamma f(X)\, \omega(P, dX).$$

It can be shown that the potential $W(P)$ is harmonic for $P \notin \Gamma$. In the proof one uses the fact that the mean-value theorem is valid for the potential $W(P)$.

The limiting values of $W(P)$ as $P \to S$, $S \in \Gamma$, from inside or from outside exist and are equal, respectively, to

$$W^{(i)}(S) = f(S) + \frac{1}{2\pi} \int_\Gamma f(X)\, \omega(S, dX),$$

$$W^{(e)}(S) = -f(S) + \frac{1}{2\pi} \int_\Gamma f(X)\, \omega(S, dX).$$

Just as in the work of I. Radon (^{3}), the exterior and interior Dirichlet problems reduce to the functional equations

$$W^{(i)} = f + Tf, \qquad W^{(e)} = -f + Tf,$$

where the operator

$$(Tf)_S = \frac{1}{2\pi} \int_\Gamma f(X)\, \omega(S, dX)$$

acts in the space $C(\Gamma)$;

$$\|T\| = \frac{1}{2\pi} \sup_{S \in \Gamma} \int_\Gamma |\omega(S, dX)| \leqslant \frac{M}{2\pi} + 1.$$

By repeating Radon' s arguments for the operator T, it is proved that the Fredholm radius of the operator T is equal to

$$2\pi \inf_{S \in \Gamma} \frac{1}{|\omega(S, S)|}.$$

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[5] L. Cesari, *Surface Area*, Princeton, 1956, p. 6.

Note: Figure translations are in progress. See original paper for figures.