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Remarks on Projective Embeddings of Algebraic Varieties

1962

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Abstract

Full Text

Mathematics

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Remarks on Projective Embeddings of Algebraic Varieties

(Presented by Academician P. S. Novikov on 27 III 1962)

1. It is known ⁽¹⁾ that to a fiber space of lines over an algebraic or complex-analytic variety there corresponds a mapping of the variety into projective space. Particularly remarkable are those fiber spaces of lines for which a sufficiently high multiple corresponds to a biregular embedding of the variety into projective space, and these high multiples themselves turn out to be induced under this embedding by a fibration over the projective space corresponding to a hyperplane. We shall call such fiber spaces of lines **nondegenerate**.

Kodaira's theorem ⁽¹⁾ gives a condition for nondegeneracy of a fiber space of complex lines over a compact complex-analytic variety.

In ⁽²⁾ we indicated the following criterion for nondegeneracy of a fiber space of lines over a nonsingular complete algebraic variety:

If V is a nonsingular complete algebraic variety, $\dim V = s$, and $[D]$ is a fiber space of lines over it, then in order that $[D]$ be nondegenerate, it is necessary and sufficient that the following conditions hold:

$$D^s > 0, \quad D^i \cdot C_i > 0, \quad i = 1, \dots, s - 1,$$

where D denotes the divisor on V corresponding to the fiber space $[D]$, and C_i runs through all i -dimensional irreducible subvarieties of the variety V .

This criterion can be generalized to arbitrary complete varieties. To this end, for every irreducible complete variety W and fiber space of lines $[D]_W$ over it, we define a number $g_W[D]_W$, which we call the degree of the fiber space of lines $[D]_W$ over W . If W is a subvariety of a nonsingular complete variety V , $\dim W = r$, and $[D]_W$ is induced by the fiber space $[D]_V$ over V , then $g_W[D]_W$ coincides with the symbol $D^r \cdot W$, defined in a natural way, where D is the divisor on V corresponding to $[D]_V$. We shall assume that the ground field k is algebraically closed of arbitrary characteristic.

Theorem 1. *Let V be a complete algebraic variety, and let $[D]_V$ be a fiber space of lines over V .*

In order that $[D]_V$ be nondegenerate, it is necessary and sufficient that the conditions

$$g_W[D]_W > 0,$$

hold, where W runs through all irreducible subvarieties of the variety V (not necessarily proper; that is, if V is irreducible, then one must also require the condition $g_V[D]_V > 0$), and $[D]_W$ is the fibration over W induced by the fibration $[D]_V$.

From this follows the following criterion for the projectivity of complete algebraic varieties:

Let V be a complete algebraic variety. In order that V be projective, it is necessary and sufficient that there exist on V a fibred space of lines $[D]_V$ satisfying the conditions

$$g_W[D]_W > 0,$$

where W runs through all irreducible (not necessarily proper) subvarieties of the variety V .

The number $g_W[D]_W$ is defined as follows. Let $\{U^{(i)}\}$ and $\{R_{ij}\}$ be the covering and transition functions on W by which $[D]_W$ is given. There always exists a system of functions $R_i \in k(W)$ such that $R_i = R_{ij}R_j$ (for example, this is $\{R_{i1}\}$). Let $\dim W = r$, and let C be an irreducible $(r-1)$ -dimensional subvariety on W , $C \cap U^{(i)} \neq \emptyset$. Represent R_i in the form $R_i = F_1/F_2$, where $F_1, F_2 \in O_C$ (O_C is the local ring of the subvariety C). Let $n_C(F)$ be the multiplicity of the ideal (F) in O_C . Put

$$n_C^{(i)}[D] = n_C(F_1) - n_C(F_2).$$

It is easy to show that if $C \cap U^{(i)} \cap U^{(j)} \neq \emptyset$, then $n_C^{(i)}[D] = n_C^{(j)}[D]$. We obtain the number

$$n_C[D] = n_C^{(i)}[D] = n_C^{(j)}[D] = \dots$$

We define $g_W[D]_W$ inductively: for $r = 0$ we set $g_W[D]_W = 1$. Suppose this number has been defined for dimensions less than r . We put

$$g_W[D]_W = \sum_C n_C[D] g_C[D]_C.$$

Here \sum_C denotes summation over all irreducible $(r-1)$ -dimensional subvarieties of the variety W (it is easy to see that in \sum_C there are only finitely many

summands different from zero); $[D]_C$ is the fibred space of lines over C induced by the fibration $[D]_W$. $g_W[D]_W$ depends only on W and $[D]_W$ (and not, for example, on the system $\{R_i\}$). This follows from the following theorem:

Theorem 2. The Euler characteristic $\chi(W, O_W[nD]_W)$ of the variety W , $\dim W = r$, with coefficients in the sheaf $O_W[nD]_W$ of local sections of the fibred space $[nD]_W$, as a function of n , is a polynomial whose leading term is equal to

$$\frac{n^r}{r!} g_W[D]_W.$$

The proof of this theorem uses the theory of normalization, as well as the following lemma, which is a generalization of a proposition proved by Serre in ⁽³⁾ for projective space.

Lemma. Let V be a complete algebraic variety; $[D]_V$ a fibred space of lines over V ; K a coherent sheaf on V , whose support has dimension s .

Then the Euler characteristic $\chi(V, K \otimes O_V[nD]_V)$ of the variety V with coefficients in the sheaf $K \otimes O_V[nD]_V$, as a function of n , is a polynomial whose degree does not exceed s .

2. We indicate some applications of Theorem 1.

A theorem of Chow ⁽⁴⁾ is known, asserting that every homogeneous variety is projective. The following theorem is true, close in its content to Chow's theorem.

Theorem 3. Let V be a complete nonsingular algebraic variety over the field of complex numbers \mathbf{C} . Let a connected algebraic group G act on V , and suppose that there exists on V an open set U_1 , which is a quasiprojective variety, such that the family of open sets $\{gU_1\}$, where g runs through all elements of the group G , covers all of V . Then V is a projective variety.

The assumption in this theorem, as well as in the following theorem, that the ground field is \mathbf{C} , is connected only with the fact that intersection theory for classes of algebraic equivalence has been constructed only for quasiprojective varieties, while in the case of the complex field one can use homology theory.

Let V be a 3-dimensional nonsingular complete variety over \mathbf{C} . We shall say that an irreducible curve d on V is absolutely immovable if, for no positive integer h , can the cycle hd be included in an infinite algebraic system of curves on V . If it is assumed that on V there are only finitely many absolutely immovable curves, then in this particular case one can answer affirmatively the following question, posed by Chevalley ⁽⁵⁾.

Suppose that a normal variety V satisfies the following condition: for any finite set of points of V there exists an affine open set on V containing all these points. Can V then be embedded in a projective space?

Theorem 4. Let V be a 3-dimensional nonsingular complete algebraic variety on which there are only finitely many absolutely immovable curves. Suppose that for any finite set of points of V there exists an affine open set on V containing all these points. Then V is a projective variety.

Let us make one more small remark concerning Nagata's constructed example of a 3-dimensional complete nonprojective variety ⁽⁵⁾. Nagata observes that the variety V^{**} constructed by him admits a covering by two quasiprojective varieties $V^{**} - l_1$ and $V^{**} - l_2$, where l_1 and l_2 are two nonintersecting projective lines on V^{**} . Let $|H_1|$ and $|H_2|$ be, respectively, the systems of hyperplane sections of the varieties $V^{**} - l_1$ and $V^{**} - l_2$. They correspond to certain linear systems $|\overline{H}_1|$ and $|\overline{H}_2|$ on V^{**} . It turns out that $\overline{H}_1 \cdot l_1 < 0$. Indeed, if $\overline{H}_1 \cdot l_1 \geq 0$, then we would obtain that the linear system on V^{**}

$$|D| = |m\overline{H}_1 + \overline{H}_2|$$

for sufficiently large m is an infinite linear system without fixed components, whose intersection with all curves on V^{**} is positive. From Theorem 1 it would then follow that V^{**} is a projective variety. $V^{**} - l_2$ can be embedded as an open subset (in the sense of the Zariski topology) in a nonsingular projective variety V' .

\overline{H}_1 defines a certain linear system H'_1 on $V^{**} - l_2$. It is easy to show that on V' there exists an infinite linear system without fixed components H' such that its intersection with $V^{**} - l_2$ coincides with H'_1 .

Since l_1 lies entirely inside the open set $V^{**} - l_2$, the ruled space of lines over V' corresponding to the linear system H' induces over l_1 the same ruled space of lines as the ruled space over V^{**} corresponding to the system \overline{H}_1 . Since $\overline{H}_1 \cdot l_1 < 0$, the degree of this ruled space over l_1 is negative, and consequently $H' \cdot l_1 < 0$.

We obtain that there exists a 3-dimensional nonsingular projective variety on which there is an infinite linear system without fixed components and an irreducible curve such that the intersection of this linear system and the curve is negative. This shows, in particular, that the following assertion of Baldassarri ⁽⁶⁾ is false: for any complete linear

system L without fixed components on a nonsingular projective variety V , for which $\dim L \geq 1$, the complete system $|nL|$ has no base points for sufficiently large n .

Received
20 III 1962

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Note: Figure translations are in progress. See original paper for figures.

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