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Abstract

Full Text

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ON CONDITIONS FOR THE EXISTENCE OF WAVE OPERATORS

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1°. Let us denote by A_1, A_2 self-adjoint operators in the Hilbert space $\mathcal{H}$; $E_\lambda^{(k)}$, $k = 1, 2$, the resolution of the identity of the operator A_k; $R_\zeta^{(k)}$ its resolvent; $D(A_k)$ the domain of definition of A_k. Let $\mathcal{H}_k$ be the set of elements $f \in \mathcal{H}$ for which the function $(E_\lambda^{(k)} f, f)$ is absolutely continuous. The set $\mathcal{H}_k$ is a subspace in $\mathcal{H}$ and reduces the operator A_k, $k = 1, 2$. The part of the operator A_k acting in $\mathcal{H}_k$ is called its absolutely continuous part. We denote the projector onto $\mathcal{H}_k$ by P_k.

In papers ([1–3]) the following assertion was established*.

Theorem 1. *Let the operator $V = A_2 - A_1$ have finite absolute trace (be a nuclear operator). Then:*

1) The strong limits exist

$$U_\pm(A_2, A_1) = \lim_{t \to \pm\infty} \exp(iA_2 t) \exp(-iA_1 t) P_1; \tag{1}$$

$$U_\pm(A_1, A_2) = \lim_{t \to \pm\infty} \exp(iA_1 t) \exp(-iA_2 t) P_2. \tag{2}$$

2) The operators $U_\pm(A_2, A_1)$ isometrically map $\mathcal{H}_1$ onto $\mathcal{H}_2$, and the operators $U_\pm(A_1, A_2) = U_\pm^*(A_2, A_1)$ isometrically map $\mathcal{H}_2$ onto $\mathcal{H}_1$.

3) The absolutely continuous parts of the operators A_1, A_2 are unitarily equivalent, and

$$U_\pm(A_2, A_1) A_1 P_1 = A_2 P_2 U_\pm(A_2, A_1).$$

This theorem, in particular, guarantees the preservation of the absolutely continuous part of the spectrum under nuclear perturbations. In applications to the spectral theory of differential equations, one usually manages to verify the conditions of Theorem 1 not for the operators A_1, A_2 themselves, but for some functions of them, for example the operators A_1^{-n}, A_2^{-n}. This is sufficient to establish the unitary equivalence of the absolutely continuous parts of the operators A_1, A_2 ([4,5]). From Theorem 1, however, there does not follow the existence of the limits (1), (2) (the wave operators) for the operators A_k themselves. At the same time the existence of such limits, arising, in particular, in problems of quantum scattering, is of independent interest. In connection with this, Kuroda ([6]) showed that in Theorem 1 the condition that the operator $V = A_2 - A_1$ be nuclear can be replaced by one of the following conditions.

a) V is symmetric on $D(A_1)$, and $\|Vf\| \leq \alpha\|A_1 f\| + \beta\|f\|$, $0 \leq \alpha < 1$, for all $f \in D(A_1)$. (Then the operator A_2 is self-adjoint on $D(A_2) = D(A_1)$.) The operator $|V|^{1/2}(A_1 - i)^{-1}$ has finite absolute norm (is a Hilbert-Schmidt operator).

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* In its full scope this assertion was established in paper ([3]).

b) A_1, A_2 are positive-definite, moreover $D(A_1^{1/2}) = D(A_2^{1/2})$, and the quadratic form of the perturbation $V[f,f] = \|A_2^{1/2} f\|^2 - \|A_1^{1/2} f\|^2$ generates a nuclear operator in the space with the metric determined by the quadratic form $\|A_1^{1/2} f\|^2$.

These assertions were established by Kuroda with the aid of Kato' s proposed [3] approximation of the perturbation operator by finite-dimensional operators. Obtaining further generalizations along this path is very difficult. At the same time, the use of Kuroda' s results in applications is limited by two reasons. First, the conditions $D(A_1) = D(A_2)$ or $D(A_1^{1/2}) = D(A_2^{1/2})$ are not fulfilled in a number of important cases, for example in problems on perturbation of the boundary and of boundary conditions [4,5]. Second, Kuroda' s conditions are certainly not fulfilled for operators of elliptic boundary-value problems in the case of a sufficiently large number of independent variables*.

Further generalizations of Theorem 1 can be obtained by discovering a connection between the operators $U_\pm(A_2, A_1)$ and $U_\pm(A_2^s, A_1^s)$. Such a connection, apparently, is also of independent interest. We note that assertions 2), 3) of Theorem 1 follow directly only from the fact of the existence of the strong limits (1), (2) (see, for example, [7]). This will allow us to confine ourselves merely to indicating conditions for the existence of the strong limits (1), (2).

Theorem 2. *Suppose there exist bounded operators A_1^{-1}, A_2^{-1}, and the operator $A_2^{-1} - A_1^{-1}$ is nuclear. Then the strong limits (1), (2) exist, and moreover*

$$U_\pm(A_2, A_1) = U_\mp(A_2^{-1}, A_1^{-1}), \qquad U_\pm(A_1, A_2) = U_\mp(A_1^{-1}, A_2^{-1}).$$

It is easy to see that Theorem 2 contains both of Kuroda's results, provided only that the operator A_1 has a real regular point. Theorem 2 is already applicable to problems on a change of the boundary, but only in the case of two independent variables (for equations of second order). In the general case, the possibility of investigating such problems is based on the following theorem.

Theorem 3. *Let the operators A_1, A_2 be positive-definite, the operator $A_2^{-1} - A_1^{-1}$ completely continuous, and the operator $A_2^{-n} - A_1^{-n}$ nuclear for some integer*

$n \geqslant 1$. *Then the strong limits (1), (2) exist, as do the limits* $U_\pm(A_2^{-1}, A_1^{-1})$, $U_\pm(A_1^{-1}, A_2^{-1})$, *and moreover*

$$U_\pm(A_k, A_l) = U_\mp(A_k^{-1}, A_l^{-1}) = U_\mp(A_k^{-n}, A_l^{-n}), \qquad k, l = 1, 2;\; k \neq l.$$

2°. We give an example of the application of Theorems 2 and 3.

In the m-dimensional Euclidean space E_m ($m \geqslant 2$) consider the self-adjoint elliptic differential expression

$$\mathcal{L}u = -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} + c(x)u. \tag{3}$$

The coefficients $a_{ij}(x)$ are continuously differentiable, the function $c(x) \geqslant 1$ is measurable and bounded in every ball. Let Ω be the exterior of a bounded domain in E_m with piecewise twice continuously differentiable boundary Γ. The coincidence of Ω with E_m is not excluded. Suppose that on a part Γ_2 of the boundary Γ a bounded measurable function $\sigma(x)$ is prescribed. In the space $L_2(\Omega)$ we introduce for consideration the semibounded quadratic form

$$\int_\Omega \left(\sum_{i,j=1}^{m} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \overline{u}}{\partial x_j} + c|u|^2 \right) dx + \int_{\Gamma_2} \sigma |u|^2 \, ds, \tag{4}$$

* This is connected with the order of the singularity of the fundamental singular solution of the corresponding elliptic equation.

obtained by closure from the initial set of continuously differentiable functions, finite in a neighborhood of infinity and of the surface $\Gamma_1 = \Gamma - \Gamma_2$. The form (4) generates in $L_2(\Omega)$ a semibounded self-adjoint operator S. This operator is given by the differential expression (3) on functions satisfying, in a certain sense, the boundary conditions

$$u\big|_{\Gamma_1} = 0, \qquad \frac{\partial u}{\partial \nu} + \sigma(x)u\big|_{\Gamma_2} = 0; \tag{5}$$

here $\partial u/\partial \nu$ is the conormal derivative. By adding to $c(x)$ a sufficiently large constant, one can make the operator S positive definite. Let A be the orthogonal sum in $L_2(E_m)$ of the operator S in $L_2(\Omega)$ and the operator of the first boundary-value problem for the differential expression (3) in $L_2(E_m - \overline{\Omega})$. Let A_1, A_2 be two operators of the indicated type, corresponding to different domains Ω and to different conditions of the form (5), with one and the same expression (3). A simple comparison of Theorems 2, 3 and the results of papers ([4],[5]) makes it possible to establish the following theorem.

Theorem 4. *In the case $m = 2$, all assertions of Theorems 1 and 2 hold for the operators A_1, A_2. In the case $m > 2$, the assertions of Theorems 1* *and 3 hold for the operators A_1, A_2, provided only that the coefficients of the differential expression (3) satisfy certain additional smoothness conditions* * *in a neighborhood of the boundary* Γ.

Using the results of papers ([4,5]), it is easy to obtain analogous assertions in other problems of a similar type. We shall not go into details on this matter here.

3°. Let us dwell briefly on the method of proof of Theorem 2. Let $\widetilde{R}_{\zeta}^{(k)}$ be the resolvent and $\widetilde{E}_{\lambda}^{(k)}$ the resolution of the identity of the operator $B_k = A_k^{-1}$, $k = 1, 2$, and let α_s, ω_s, $s = 1, 2, ...$, be the eigenvalues and eigenelements of the operator $B_2 - B_1$. Using formula (1)

$$\exp(itB_2)\exp(-itB_1) = E + i\int_0^t \exp(i\tau B_2)(B_2 - B_1)\exp(-i\tau B_1)\,d\tau \qquad (6)$$

and passing in (6) to the weak Abel limit as $t \to -\infty$, one can show that

$$(U_{-}(B_2, B_1)f, g) =$$

$$= (P_1 f, P_2 g) + \lim_{\varepsilon\to+0}\sum_{s=1}^{\infty}\alpha_s\int_{-\infty}^{+\infty}(\widetilde{R}_{\mu-i\varepsilon}^{(1)}P_1 f, \omega_s)\frac{d}{d\mu}(\widetilde{E}_{\mu}^{(2)}\omega_s, g)\,d\mu. \qquad (7)$$

The expression $(\widetilde{R}_{\mu-i\varepsilon}^{(1)}P_1 f, \omega_s)$ as $\varepsilon \to +0$ has a finite limit for almost all μ. In expression (7) one can carry out the limiting passage as $\varepsilon \to +0$ on dense sets of elements f and g. With the aid of the relation

$$U_t = \exp(itA_2)\exp(-itA_1) =$$

$$= E - i\int_0^t \exp(itA_2)\,A_2(B_2 - B_1)A_1\exp(-itA_1)\,d\tau$$

one can, for the weak limit of $P_2 U_{tP} 1$ as $t \to +\infty$, obtain the expression

$$(P_1 f, P_2 g) - \lim_{\varepsilon\to+0}\sum_{s=1}^{\infty}\alpha_s\int_{-\infty}^{+\infty}(R_{\nu+i\varepsilon}^{(1)}P_1 A_1 f, \omega_s)\frac{d}{d\nu}(E_{\nu}^{(2)}\omega_s, P_2 A_2 g)\,d\nu. \qquad (8)$$

* For the smoothness conditions, see ([4],[5]).

In expression (8) one may also pass to the limit as $\varepsilon \to +0$ on the same dense sets f and g. Using the relations

$$\widetilde{E}_\lambda^{(2)} = E_0^{(2)} - E_{\frac{1}{\lambda}+0}^{(2)} \quad (\lambda \leqslant 0); \qquad \widetilde{E}_\lambda^{(2)} = E + E_0^{(2)} - E_{\frac{1}{\lambda}+0}^{(2)} \quad (\lambda \geqslant 0);$$

$$\widetilde{R}_\zeta^{(1)} = -\frac{1}{\zeta} R_{1/\zeta}^{(1)} A_1,$$

it is easy to establish the coincidence of expressions (7) and (8) (after passing in them to the limit as $\varepsilon \to +0$). The existence, as $t \to +\infty$, of the strong limit (1) now follows from the fact that the weak limit of the operator $P_2 U_{tP} 1$ coincides with the isometric operator $U_-(B_2, B_1)$.

The proof of Theorem 3 is based on considerations of the same nature, but is technically considerably more complicated.

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References Cited

[1] M. Rosenblum, *Pacific J. Math.*, **7**, No. 1 (1957).
[2] T. Kato, *J. Math. Soc. Japan*, **9**, No. 2 (1957).
[3] T. Kato, *Proc. Japan Acad.*, **33**, No. 5 (1957).
[4] M. Sh. Birman, *DAN*, **137**, No. 4 (1961).
[5] M. Sh. Birman, *Vestn. LGU*, No. 1 (1962).
[6] S. T. Kuroda, *J. Math. Soc. Japan*, I, **11**, No. 3 (1959); II, **12**, No. 3 (1960).
[7] S. T. Kuroda, *Nuovo Cim.*, **12**, No. 5 (1959).

Note: Figure translations are in progress. See original paper for figures.