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Abstract

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ONE EXAMPLE IN THE THEORY OF EMBEDDING THEOREMS*

(Presented by Academician S. L. Sobolev, 27 XI 1961)

Let $W_p^{(r)}(D)$ $(p > 1,\ r > 0)$ denote the class of functions $f(x) = f(x_1, \dots, x_n)$, defined on the rectangle $D = \{x : 0 < x_i < 1,\ i = 1, \dots, n\}$, which have there all generalized partial derivatives of orders $1, 2, \dots, [r]$ and finite norm

$$\|f\|_{W_p^{(r)}(D)} = \|f\|_{L_p(D)} + \|f\|_{L_p^{(r)}(D)},$$

where

$$\|f\|_{L_p^{(r)}(D)} = \sum_{i_1,\dots,i_r=1}^{n} \left\| \frac{\partial^r f}{\partial x_{i_1} \cdots \partial x_{i_r}} \right\|_{L_p(D)}$$

when r is an integer,

$$\|f\|_{L_p^{(r)}(D)} = \sum_{i_1,\dots,i_{[r]}=1}^{n} \left\{ \iint_{D\ D} \frac{\left| \frac{\partial^{[r]} f(x)}{\partial x_{i_1} \cdots \partial x_{i_{[r]}}} - \frac{\partial^{[r]} f(y)}{\partial x_{i_1} \cdots \partial x_{i_{[r]}}} \right|^p}{|x-y|^{n+(r-[r])p}} \, dx\, dy \right\}^{1/p}$$

when r is not an integer.

The spaces $W_p^{(r)}$ for integral r were studied by S. L. Sobolev, and for arbitrary $r > 0$ by L. N. Slobodetskii.

Put $\mu = r - n/p$. We shall consider all possible pairs of numbers (μ, r) for which the given function f turns out to belong to $W_p^{(r)}(D)$. Such points form a certain set $\Xi^*(f)$. We denote the set of its interior points by $\Xi(f)$. The sets $\Xi^*(f)$, $\Xi(f)$ are contained in the half-strip

$$\Lambda = \{(\mu, r) : r > 0,\ \mu < r < \mu + n\}.$$

From the embedding theorems for classes of functions, and also from theorems of the Gagliardo-Nirenberg type on the embedding of the intersection of two classes into a third, it follows that the sets $\Xi^*(f)$ and $\Xi(f)$ occupy the lower part of the half-strip Λ, separated from its upper part by a nondecreasing convex curve.

S. L. Sobolev put forward the hypothesis that any admissible domain of the half-strip Λ is the domain $\Xi(f)$ for some function $f(x)$. In the present note we give a proof of this hypothesis. Namely, the following holds.

Theorem. *Let the domain H, which is the lower part of the half-strip Λ, be separated from the upper part by a nondecreasing convex curve $r = r(\mu)$. Then there exists a function $f(x)$ for which the set $\Xi(f)$ coincides with H.*

We first prove a lemma.

Lemma. *The theorem is true in the particular case when $r = r(\mu)$ is a straight line whose angular coefficient differs from $-\dfrac{m}{n-m}$.*

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* The result of this note was reported at the IV All-Union Mathematical Congress in July 1961.

Proof. Let $\omega_m(x)$ be a smooth function depending only on $x_1, \dots, x_m$ $(1 \le m \le n)$, with support contained in the set

$$D_m = \{x : 0 < x_i < 1,\ i = 1, 2, \dots, m\}.$$

Consider the function

$$f(x) = \sum_{k=k_0}^{\infty} k^{-\beta} \omega_m \left[k^{\delta} \left(x - \frac{e_1}{\ln k} \right) \right] = \sum_{k=k_0}^{\infty} \Omega_k(x).$$

Here $e_1 = (1, 0, \dots, 0)$, $\delta > 1$, and $k_0 = k_0(\lambda)$ is chosen so large that the supports of distinct $\Omega_k(x)$ do not overlap. Let us find the set $\Xi^*(f)$.

By changes of variables it is easy to see that

$$\|\Omega_k\|_{L_p^{(r)}(D)} \sim k^{\delta(r-m/p)}.$$

and also that

$$\|f(x)\|^p_{L_p^{(r)}(D)} \sim \sum_{k=k_0}^{\infty} k^{-\beta p + \delta(rp-m)}.$$

The last series converges if and only if $-\beta p + \delta(rp - m) < -1$, or

$$[\delta(n-m)+1]r + (\delta m - 1)\mu - \beta n < 0. \tag{1}$$

Thus we see that the set $\Xi^*(f)$ consists of the points lying below the line (1). By choosing m, $\delta > 1$, β, this line can assume any admissible position except one parallel to the vectors $(m-n, m)$, where $m = 0, 1, \dots, n$. The lemma is thereby proved.

Proof of the theorem. Suppose now that an admissible domain H is given, representing the lower part of the half-plane Λ, separated from the upper part by a convex nondecreasing curve $r = r(\mu)$. One can indicate a countable number of lines l_i with negative angular coefficients (not equal, however, to $-\frac{m}{n-m}$, where $m = 1, \dots, n$), cutting off from below, from the half-plane Λ, sets Δ_i such that the set of interior points

$$\bigcap_{i=1}^{\infty} \Delta_i$$

coincides with H. With each line l_i let us associate, by the method indicated in the lemma, the function

$$f_i(x) = \sum_{k=k_i}^{\infty} k^{\beta_i} \omega_{m_i} \left[k^{\beta_i} \left(x - \frac{1}{\ln k} - 2^{-i} \right) \right].$$

We choose β_i, δ_i, and m_i in such a way that the equation

$$[\delta_i(n - m_i) + 1]r + (\delta_i m_i - 1)\mu - \beta_i n = 0 \tag{2}$$

is the equation of the line l_i. Suppose also that the numbers k_i are chosen so large that

$$2^{-i} + \ln^{-1} k < 2^{-i+1} \quad \text{for } k \geq k_i \quad (i = 1, 2, \dots).$$

This ensures that the supports of two distinct functions f_i and f_j do not intersect. Consider now the function

$$f(x) = \sum_{1}^{\infty} f_i(x). \tag{3}$$

Whatever point ξ of the half-plane Λ may be, $\xi \notin \overline{H}$, there exists a line l_i situated below it. In view of the nonintersection of the supports of the terms of the series (2)

one may assert that $\xi \in \Xi^*(f)$. We shall now show that, for a special choice of the lines l_i and the numbers k_i, $\Xi^*(f) = \text{H}$. Put

$$-\eta\gamma_i = [\delta_i(n - m_i) + 1]r + (\delta_i m_i - 1)\mu - \beta_i n < 0.$$

Let us note that, for fixed δ_i and m_i, the distance from the point (r, μ) to the line (2) is proportional to γ_i and depends on β_i and the point (r, μ). Let $\sigma_i = \inf \gamma_i$ for $(r, \mu) \in \mathrm{H}$. The lines l_i can always be chosen so that $\sigma_i = \sigma_i(\delta_i, m_i, \beta_i)$ tend to zero sufficiently slowly. In this case, by means of estimates analogous to those indicated in the lemma, we shall have:

$$\|f\|^p_{L^r_p(D)} \leq A(\mu, p) \sum_{i=1}^{\infty} \sum_{k=k_i}^{\infty} k^{-1-p\gamma_i} \leq$$

$$\leq A_1(\mu, p) \sum_{i=1}^{\infty} \frac{1}{p\gamma_i k_i^{p\gamma_i}} \leq A_1(\mu, p) \sum_{i=1}^{\infty} \frac{1}{\sigma_i k_i^{\sigma_i}}.$$

Here σ_i does not depend on the point $(r, \mu) \in \mathrm{H}$, and therefore it is possible to specify a sequence k_i for which the last series converges. Thus the proof is complete.

Remark. Denote by $\Xi^{**}(f)$ the set of points of $\Xi(f)$ and the curve $r = r(\mu)$ (separating the set $\Xi(f)$ from the upper part of the half-plane Λ). By the same method it is shown that an arbitrary admissible set H^{**} is the set $\Xi^{**}(f)$ for some function $f(x)$.

It would be interesting to establish whether every (admissible) set H^* is the set $\Xi^*(f)$ for some function $f(x)$.

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Note: Figure translations are in progress. See original paper for figures.

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