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Abstract

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MATHEMATICS

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ON THE HILBERT BOUNDARY-VALUE PROBLEM WITH SHIFT

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Take a closed Lyapunov contour L, bounding a finite domain D^+, and consider the following boundary-value problem. Find a function $\Phi(z) = u(x,y)+iv(x,y)$, analytic in D^+ and continuous in $D^+ + L$, whose boundary values of the real and imaginary parts satisfy on L the condition

$$a(t)u(t) + b(t)u[\alpha(t)] + c(t)v(t) + d(t)v[\alpha(t)] = h(t). \tag{1}$$

Here $a(t), b(t), c(t), d(t), h(t)$ are real functions of points of the contour L, satisfying on L a Hölder condition; the function $\alpha(t)$ has a Hölder-continuous derivative $\alpha'(t) \neq 0$, maps the contour L homeomorphically onto itself, and satisfies the Carleman condition (1) $\alpha[\alpha(t)] \equiv t$.

The boundary condition (1) may be written in the form

$$A(t)\Phi^+(t) + \overline{A(t)}\,\overline{\Phi^+(t)} + B(t)\Phi^+[\alpha(t)] + \overline{B(t)}\,\overline{\Phi^+[\alpha(t)]} = h(t), \tag{1'}$$

where $A(t) = a(t) - ic(t)$, $B(t) = b(t) - id(t)$. We shall seek the solution of the boundary-value problem $(1')$ in the form (2)

$$\Phi(z) = \frac{1}{\pi i}\int_L \frac{\mu(\tau)}{\tau - z}\,d\tau + iC, \tag{2}$$

where $\mu(t)$ is a real function and C is a certain real constant. Substituting into $(1')$ the limiting value of the Cauchy-type integral (2), we arrive at the integral equation:

$$[A(t)+\overline{A(t)}]\mu(t)+[B(t)+\overline{B(t)}]\mu[\alpha(t)]+\frac{1}{\pi i}\int_L\left[\frac{A(t)}{\tau-t}-\frac{\overline{A(t)}\,\overline{\tau}^2}{\overline{\tau}-\overline{t}}\right]\mu(\tau)\,d\tau+$$

$$+\frac{\lambda}{\pi i}\int_K\left[\frac{B(t)\alpha'(\tau)}{\alpha(\tau)-\alpha(t)}-\frac{\overline{B(t)}\,\overline{\alpha'(\tau)}\,\overline{\tau}^2}{\overline{\alpha(\tau)}-\overline{\alpha(t)}}\right]\mu[\alpha(\tau)]\,d\tau=g(t), \tag{3}$$

where $g(t)=h(t)+2C\ \mathrm{Im}[A(t)+B(t)]$, $\lambda=1$ or $\lambda=-1$, according as the transformation $\alpha(t)$ preserves or reverses the direction of traversal on L.

Consider the following system of singular integral equations:

$$[A(t)+\overline{A(t)}]\varphi(t)[B(t)+\overline{B(t)}]\psi(t)+\frac{1}{\pi i}\int_L\left[\frac{A(t)}{\tau-t}-\frac{\overline{A(t)}\,\overline{\tau}^2}{\overline{\tau}-\overline{t}}\right]\varphi(\tau)\,d\tau+$$

$$+\frac{\lambda}{\pi i}\int_L\left[\frac{B(t)\alpha'(\tau)}{\alpha(\tau)-\alpha(t)}-\frac{\overline{B(t)}\,\overline{\alpha'(\tau)}\,\overline{\tau}^2}{\overline{\alpha(\tau)}-\overline{\alpha(t)}}\right]\psi(\tau)\,d\tau=g(t),$$

$$[B[\alpha(t)]+\overline{B[\alpha(t)]}]\varphi(t)+[A[\alpha(t)]+\overline{A[\alpha(t)]}]\psi(t)+ \tag{4}$$

$$+\frac{1}{\pi i}\int_\tau\left[\frac{B[\alpha(t)]}{\tau-t}-\frac{\overline{B[\alpha(t)]}\,\overline{\tau}^2}{\overline{\tau}-\overline{t}}\right]\varphi(\tau)\,d\tau+$$

$$+\frac{\lambda}{\pi i}\int_0\left[\frac{A[\alpha(t)]\alpha'(\tau)}{\alpha(\tau)-\alpha(t)}-\frac{\overline{A[\alpha(t)]}\,\overline{\alpha'(\tau)}\,\overline{\tau}^2}{\overline{\alpha(\tau)}-\overline{\alpha(t)}}\right]\psi(\tau)\,d\tau=g[\alpha(t)].$$

System (4) belongs to the normal type (^{3}), if

$$(1+\lambda)\{\overline{A(t)A[\alpha(t)]}-\overline{B(t)B[\alpha(t)]}\}+$$

$$+(1-\lambda)\{\overline{A(t)}A[\alpha(t)]-B(t)\overline{B[\alpha(t)]}\}\neq 0\quad \text{on } L.$$

It is easy to see that if the nonhomogeneous equation (3) is solvable, then the nonhomogeneous system (4) is also solvable. Replacing in system (4) t by $\alpha(t)$ and τ by $\alpha(\tau)$, we obtain that, together with the solution $\varphi_1(t),\psi_1(t)$, system (4) has the solution $\varphi_2(t)=\psi_1[\alpha(t)]$, $\psi_2(t)=\varphi_1[\alpha(t)]$. Therefore, if system (4) is solvable, then there exists a solution of system (4) satisfying the condition $\varphi(t)=\psi[\alpha(t)]$, and, consequently, the function $\varphi(t)$ gives a solution of equation (3). Thus, the nonhomogeneous equation (3) is solvable if and only if the nonhomogeneous system (4) is solvable. For the solvability of the latter it is necessary and sufficient ([3]) that the condition

$$\int_L \{g(t)\rho(t) + g[\alpha(t)]\omega(t)\}\, dt = 0, \tag{5}$$

hold, where $\rho(t), \omega(t)$ is a solution of the homogeneous system of equations adjoint to system (4).

It is not difficult to show that every solution of the adjoint system satisfies one of the two conditions

$$\omega(t) - \lambda\alpha'(t)\rho[\alpha(t)] = 0 \quad \text{or} \quad \omega(t) + \lambda\alpha'(t)\rho[\alpha(t)] = 0. \tag{6}$$

If $\omega(t) + \lambda\alpha'(t)\rho[\alpha(t)] = 0$ on L, then condition (5) is satisfied automatically. If, however, on L

$$\omega(t) - \lambda\alpha'(t)\rho[\alpha(t)] = 0,$$

then $\rho(t)$ gives a solution of the equation

$$[A(t) + \overline{A(t)}]\nu(t) + \lambda[B[\alpha(t)] + \overline{B[\alpha(t)]}]\alpha'(t)\nu[\alpha(t)] -$$

$$-\frac{1}{\pi i}\int_L \left[\frac{A(\tau)}{\tau - t} - \frac{t'^2\overline{A(\tau)}}{\tau - t}\right]\nu(\tau)d\tau -$$

$$-\frac{\lambda}{\pi i}\int_L \left[\frac{B[\alpha(\tau)]}{\tau - t} - t'^2\frac{\overline{B[\alpha(\tau)]}}{\tau - t}\right]\alpha'(\tau)\nu[\alpha(\tau)]d\tau = 0, \tag{7}$$

adjoint to equation (3). Using the first condition (6), we obtain that for the solvability of equation (3) it is necessary and sufficient that the condition

$$\int_L g(t)\nu(t)\, dt = 0, \tag{8}$$

hold, where $\nu(t)$ is a solution of equation (7). Let us note that $\nu_1(t) = \nu(t)t'(s)$ is a real function. Taking into account that

$$t'(s)\big|_{\alpha(t)} = \frac{\alpha'(t)}{|\alpha'(t)|}\, t'(s),$$

equation (7) can be written in the form

$$\operatorname{Re}\{A(t)\nu_1(t) + \lambda B[\alpha(t)]|\alpha'(t)|\nu_1[\alpha(t)] -$$

$$-\frac{1}{\pi i}\int_L \frac{t'(s)A(\tau)\nu_1(\tau)}{\tau - t}\,d\tau - \frac{\lambda}{\pi i}\int_L \frac{t'(s)B[\alpha(\tau)]|\alpha'(\tau)|}{\tau - t}\nu_1[\alpha(\tau)]\,d\tau\Big\} = 0. \quad (7')$$

Condition (8) now assumes the form

$$\int_L g(t)\nu_1(t)\,ds = 0,$$

where $\nu_1(t)$ is a solution of equation $(7')$. We have proved:

Theorem 1. *The functional equation (3) is normally solvable.*

Thus, for the boundary-value problem (1) the Noether theorems are valid, asserting the finiteness of the number of linearly independent solutions of this problem and its normal solvability.

Let us construct the problem adjoint to the boundary-value problem $(1')$. For this purpose we introduce the piecewise-analytic function

$$\Psi(z) = \frac{1}{2\pi i}\int_L \frac{A(\tau)\nu_1(\tau) + \lambda B[\alpha(\tau)]|\alpha'(\tau)|\nu_1[\alpha(\tau)]}{\tau - z}\,d\sigma.$$

By virtue of equation $(7')$, $\operatorname{Re}[\overline{t'(s)}\Psi^-(t)] = 0$; hence, assuming the condition

$$\operatorname{Im}\int_L \{A(\tau)v_1(\tau) + \lambda B[\alpha(\tau)]|\alpha'(\tau)|v_1[\alpha(\tau)]\}\,d\sigma = 0,$$

we obtain that $\Psi^-(z) \equiv 0$. Then

$$[A(t)v_1(t) + \lambda B[\alpha(t)]|\alpha'(t)|v_1[\alpha(t)]]\overline{t'(s)} = \Psi^+(t). \quad (9)$$

Eliminating $v_1(t)$ from (9), we obtain the adjoint problem. In the case $\lambda = 1$ it is convenient to write the latter in the form

$$\operatorname{Re}\{i[\overline{A(t)}A[\alpha(t)] - \overline{B(t)}B[\alpha(t)]]t'(s)[A[\alpha(t)]\Psi^+(t) - \lambda B[\alpha(t)]\alpha'(t)\Psi^+[\alpha(t)]]\} = 0,$$

and in the case $\lambda = -1$ in the form

$$\operatorname{Re}\{i[A(t)\overline{A[\alpha(t)]} - \overline{B(t)}B[\alpha(t)]][\overline{A[\alpha(t)]}t'(s)\Psi^+(t) - \lambda\overline{B[\alpha(t)]}\alpha'(t)t'(s)\Psi^+[\alpha(t)]]\} = 0.$$

Under some additional conditions imposed on the coefficients of the boundary-value problem (1′), this problem can be investigated completely. Suppose that

$$B(t)\overline{A[\alpha(t)]} - \overline{B(t)}A[\alpha(t)] \neq 0 \quad \text{on } L. \tag{10}$$

Consider two cases:

I, 1. Let $\alpha(t)$ change the orientation of the contour L, i.e. $\lambda = -1$, and suppose the condition

$$A(t)A[\alpha(t)] = B(t)B[\alpha(t)]. \tag{11}$$

In the case under consideration, condition (10) is evidently the condition of normality of the system (4). Using (11), we reduce the boundary-value problem (1′) to the equivalent Carleman problem ([1],[4])

$$\Phi^+[\alpha(t)] = -\frac{A(t)}{B(t)}\Phi^+(t) + \frac{\overline{A[\alpha(t)]}h(t) - \overline{B(t)}h[\alpha(t)]}{B(t)\overline{A[\alpha(t)]} - \overline{B(t)}A[A(t)]}, \tag{12}$$

where $A(t) \neq 0,\ B(t) \neq 0$ on L by virtue of condition (10). Denote

$$\varkappa = \operatorname{Ind}\frac{a - ic}{a - id}.$$

Then, under conditions (10) and (11), the following is valid.

Theorem 2. If $\varkappa \leqslant 0$, then the homogeneous problem (1) has $-\varkappa + 1$ linearly independent solutions for odd $\varkappa$ and $-\varkappa + 1 + \gamma$ solutions for even $\varkappa$; $\gamma = 1$, if $A(t_k) = -B(t_k)$; $\gamma = -1$, if $A(t_k) = B(t_k)$; $t_k,\ k = 1, 2$, are the fixed points of the shift $\alpha(t)$. The nonhomogeneous problem (1) for $\varkappa \leqslant 0$ is unconditionally solvable. If $\varkappa > 0$, then the homogeneous problem (1) has no nontrivial solutions, and for solvability of the corresponding nonhomogeneous problem it is necessary and sufficient that $\varkappa - 1$ conditions be fulfilled for $\varkappa = 2n - 1$ and $\varkappa - 1 - \gamma$ conditions for $\varkappa = 2n$.

I, 2. Let $\alpha(t)$ preserve the orientation of L ($\lambda = 1$), and suppose the condition

$$A(t)\overline{A[\alpha(t)]} = \overline{B(t)}B[\alpha(t)]. \tag{13}$$

Condition (10), as in case I, 1, means the condition of normality of the system (4). Using (13), we obtain from (1′) a boundary-value problem of Carleman-problem type

$$\Phi^+[\alpha(t)] = -\frac{\overline{A(t)}}{B(t)}\Phi^+(t) + \frac{\overline{A[\alpha(t)]}h(t) - \overline{B(t)}h[\alpha(t)]}{\overline{A[\alpha(t)]}B(t) - A[\alpha(t)]\overline{B(t)}}. \tag{14}$$

Denote

$$\varkappa = \operatorname{Ind} \frac{a+ic}{b-id}.$$

For the study of problem (14) one can apply a method close to the method of solving problem (12) given in ([4]). In particular, one can show that $\varkappa$ can only be an even number.

Theorem 3. Under conditions (10) and (13), the homogeneous problem (1) has $\varkappa+1$ linearly independent solutions if $\varkappa \geq 0$, and has no nontrivial solutions if $\varkappa < 0$. The corresponding nonhomogeneous problem in the case $\varkappa < 0$ is solvable provided that $-\varkappa-1$ solvability conditions are satisfied, and is unconditionally solvable for $\varkappa \geq 0$.

Let us note one particular case of Theorem 3. Suppose that in the boundary condition (1) $a(t) = d(t) \equiv 0$. Then conditions (10) and (13) take the form: $b(t) \neq 0$, $c(t) \neq 0$, and $c(t)c[\alpha(t)] = b(t)b[\alpha(t)]$. We have that $\varkappa = \operatorname{Ind} \dfrac{ic(t)}{b(t)} = 0$. Consequently, the boundary-value problem $b(t)u[\alpha(t)] + c(t)v(t) = h(t)$ always has a solution depending linearly on one real arbitrary constant. This result was obtained earlier by I. M. Melnik ([5]).

II. Let now

$$B(t)\overline{A[\alpha(t)]} - \overline{B(t)}A[\alpha(t)] = 0 \quad \text{on } L. \tag{15}$$

Suppose also that on L one of the conditions is satisfied

$$A(t)\overline{A[\alpha(t)]} - \overline{B(t)}B[\alpha(t)] \neq 0 \quad \text{or} \quad \overline{A(t)}A[\alpha(t)] - B(t)\overline{B[\alpha(t)]} \neq 0. \tag{16}$$

It is easy to show, using (15), that if

$$A(t)\overline{A[\alpha(t)]} - \overline{B(t)}B[\alpha(t)] = 0,$$

then also

$$\overline{A(t)}A[\alpha(t)] - B(t)\overline{B[\alpha(t)]} = 0,$$

and conversely.

Therefore the satisfaction of one of the inequalities (16) entails the satisfaction of the other inequality. Consequently, in the case under consideration the system (4) belongs to the normal type both for $\lambda = 1$ and for $\lambda = -1$. The boundary-value problem ($1'$), in view of (15), is equivalent to the problem

$$\Phi^+(t) = -\frac{\overline{A(t)}}{A(t)}\overline{\Phi^+(t)} + \frac{\overline{A[\alpha(t)]}h(t) - \overline{B(t)}h[\alpha(t)]}{A(t)\overline{A[\alpha(t)]} - B[\alpha(t)]\overline{B(t)}}. \tag{17}$$

Denote

$$\varkappa = \operatorname{Ind}\frac{a+ic}{a-ic} = 2\operatorname{Ind}(a+ic) = 2\varkappa_1.$$

Theorem 4. The homogeneous problem (1) under conditions (15) and (16) has no nontrivial solutions if $\varkappa_1 = \operatorname{Ind}(a+ic) < 0$, and has $2\varkappa_1 + 1$ linearly independent solutions if $\varkappa_1 \geq 0$. The corresponding nonhomogeneous problem for $\varkappa_1 \geq 0$ is unconditionally solvable, while for $\varkappa_1 < 0$ there exists a unique solution of this problem provided that $-2\varkappa_1 - 1$ solvability conditions are satisfied.

Let us note that the ordinary Hilbert problem $a(t)u(t) + c(t)v(t) = h(t)$ is a particular case of problem (1) with conditions (15) and (16). Here $b(t) = d(t) = 0$; consequently, condition (15) is satisfied automatically, while condition (16) gives $a^2(t) + c^2(t) \neq 0$.

The indices $\varkappa$ of the boundary-value problems (12), (14), (17) and the indices $\varkappa^*$ of the corresponding adjoint problems are connected by the relations $\varkappa^* = -\varkappa + 2$ in case I, 1 and $\varkappa^* = -\varkappa - 2$ in cases I, 2 and II. Analysis of the adjoint problems shows that, for the boundary-value problem (1), in cases I, 1, I, 2 and II, all three Noether theorems are valid.

Analogously, one may consider the boundary-value problem (1) for the case of an infinite domain D^-.

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Note: Figure translations are in progress. See original paper for figures.