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# MATHEMATICS

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## Abstract

## Full Text

# MATHEMATICS

### S. V. Nemchinov

# On the Solution by a Non-Iterative Grid Method of Boundary-Value Problems for Partial Differential Equations with Periodic Boundary Conditions

*(Presented by Academician A. A. Dorodnitsyn, 24 V 1962)*

In paper ($^1$), solutions were considered of boundary-value problems in a rectangle for the Helmholtz equation, obtained by the grid method without the use of iterative computational techniques. If, however, it is necessary to obtain by the grid method the solution of a boundary-value problem for the Helmholtz equation, for example in a disk, then the use of a rectangular Cartesian coordinate system is associated with well-known difficulties in approximating the curvilinear boundary of the domain. Generally speaking, a grid of points with a sufficiently small step $\Delta s$ between the grid nodes can be inscribed in a disk. For such a grid domain approximating the disk, the solution of the boundary-value problem can be carried out by the method considered in paper ($^2$). However, even in this case additional difficulties arise if, for example, the Neumann boundary-value problem is considered for a disk.

It would be natural to consider the boundary-value problem in a disk for partial differential equations in polar coordinates. Then, in passing from rectangular Cartesian coordinates, the boundary-value problem for the disk in the $(x, y)$-plane becomes a boundary-value problem for a rectangle with sides $0 \leq r \leq R$, $0 \leq \varphi \leq 2\pi$ in the $(r, \varphi)$-plane. The solutions of boundary-value problems for a rectangle by the grid method considered in ($^1$) can be successfully applied to this case. The principal difficulty that must be overcome here consists in the fact that, in polar coordinates, when passing to finite differences with respect to the variable $\varphi$, it is necessary to satisfy the condition of periodicity of the solution at the points $\varphi_k$*

$$Q(r, \varphi_k) = Q(r, 2\pi + \varphi_k), \qquad \varphi_k = \frac{2\pi k}{n}, \qquad k = 0, 1, \dots, n-1.$$

Thus, finding the solution by the grid method of a boundary-value problem for a rectangle with periodic boundary conditions makes it possible to construct general algorithms for solving boundary-value problems by the grid method not only for a disk, but also for an annulus, a cylinder, a cylindrical layer, etc.

Thus, let us consider the Dirichlet and Neumann boundary-value problems for the Helmholtz equation

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial Q}{\partial r} + \frac{1}{r^2}\frac{\partial^2 Q}{\partial \varphi^2} - k^2 Q = F(r, \varphi) \tag{1}$$

with $Q(R, \varphi) = P(\varphi)$, $Q(0, \varphi)$ bounded –problem D; with $Q'_r(R, \varphi) = S(\varphi)$, $Q'_r(0, \varphi)$ bounded –problem N.

As in (1), to solve the posed boundary-value problems D and N in the rectangle $0 \leq r \leq R;\ 0 \leq \varphi \leq 2\pi$, we apply to equation (1) the method of lines with respect to the variable $\varphi$. On the interval $[0, 2\pi]$ choose the points $\varphi_j = 2\pi j/n$, $j = 0, 1, \dots, n-1$. Introduce the notation: $Q(r, \varphi_j) = q_j(r)$, $F(r, \varphi_j) = f_j(r)$, $P(\varphi_j) = p_j$, $S(\varphi_j) = s_j$. For the operator $\partial^2 Q/\partial\varphi^2$, using centered finite differences, we choose the representation[*]

$$\left.\frac{\partial^2 Q}{\partial \varphi^2}\right|_{\varphi=\varphi_j} = \frac{1}{h_1^2}\,(q_{j+1} - 2q_j + q_{j-1}), \qquad h_1 = \frac{2\pi}{n}. \tag{2}$$

Then equation (1), written on each straight line $\varphi_j$, gives a system of ordinary differential equations, whose matrix notation has the form

$$\frac{1}{r}\frac{d}{dr} r \frac{d\widetilde{Q}(r)}{dr} - \left[k^2 E + \frac{1}{r^2 h_1^2}(2E - G_n)\right]\widetilde{Q}(r) = \widetilde{F}(r). \tag{3}$$

Here

$$\widetilde{Q} = \begin{Vmatrix} q_0 \\ q_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ q_{n-1} \end{Vmatrix}, \qquad \widetilde{F} = \begin{Vmatrix} f_0 \\ f_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f_{n-1} \end{Vmatrix}, \qquad G_n = \begin{Vmatrix} 0 & 1 & 0 & \cdots & 0 & 1 & & \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 1 & 0 \end{Vmatrix},$$

$E$ is the identity matrix.

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[*] The appearance of variable coefficients in the Helmholtz differential operator when passing to polar coordinates does not introduce any fundamental difficulties, since the realization of the finite-difference analogue of the problem with respect to the variable $r$ can be carried out by the sweep method.

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It can be shown that the matrix $G_n$ has a simple structure, and for it there exists the representation

$$G_n = HLH, \qquad H' = H, \qquad H^2 = E, \tag{4}$$

where $H = \|h_{\alpha\beta}\|_n$ is the fundamental matrix of the matrix $G_n$ with elements

$$h_{\alpha\beta} = \frac{1}{\sqrt{n}}\left(\cos\frac{2\pi\alpha\beta}{n} + \sin\frac{2\pi\alpha\beta}{n}\right), \qquad \alpha, \beta = 1, 2, \dots, n \tag{5}$$

and $L = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ is a diagonal matrix with elements

$$\lambda_j = 2\cos\frac{2\pi j}{n}, \qquad j = 1, 2, \dots, n, \tag{6}$$

the eigenvalues of the matrix $G_n$. To compute the rows of the matrix $H$, the recurrence formula

$$h_{\alpha,\gamma+2} = \lambda_\alpha h_{\alpha,\gamma+1} - h_{\alpha,\gamma}, \qquad \gamma = 0, 1, \dots, n-2, \tag{7}$$

may be used, with

$$h_{\alpha,0} = h_{\alpha,n} = \frac{1}{\sqrt{n}}, \qquad h_{\alpha,1} = \frac{1}{\sqrt{n}}\left(\cos\frac{2\pi\alpha}{n} + \sin\frac{2\pi\alpha}{n}\right).$$

If we now introduce into consideration the vectors

$$H\widetilde{Q} = \overline{Q} = \left\|\begin{matrix} \overline{q}_0 \\ \overline{q}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \overline{q}_{n-1} \end{matrix}\right\|, \qquad H\widetilde{F} = \overline{F} = \left\|\begin{matrix} \overline{f}_0 \\ \overline{f}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \overline{f}_{n-1} \end{matrix}\right\|, \tag{8}$$

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* Obviously, in formula (2) the periodicity condition in $\varphi$ for the desired solution requires satisfaction of the equalities

$$q_{k_j} \equiv Q(r, \varphi_k) = Q(r, 2\pi + \varphi_k) = Q(r, \varphi_{k+n}) = q_{k+n}, \qquad n = -1, 0, 1.$$

The special form of the matrix $G_n$ takes this circumstance into account.

$$\overline{P} = H \left\| \begin{matrix} p_0 \\ p_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ p_{n-1} \end{matrix} \right\| = \left\| \begin{matrix} \overline{p}_0 \\ \overline{p}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \overline{p}_{n-1} \end{matrix} \right\|, \qquad \overline{S} = H \left\| \begin{matrix} s_0 \\ s_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ s_{n-1} \end{matrix} \right\| = \left\| \begin{matrix} \overline{s}_0 \\ \overline{s}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \overline{s}_{n-1} \end{matrix} \right\|. \tag{8}$$

and multiply equation (3) on the left by the matrix $H$, while for the vector-matrix $\overline{Q}$ we obtain the equation

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\overline{Q}(r) - \left[k^2 E + \frac{1}{r^2 h_1^2}(2E + G_n)\right]\overline{Q}(r) = \overline{F}(r), \tag{9}$$

which, for the components of the vector-matrix $\overline{Q}$, represents $n$ independent second-order equations with variable coefficients of the form

$$\frac{1}{r}\frac{d}{dr}r\frac{d\overline{q}_j}{dr} - \left[k^2 + \frac{2-\lambda_j}{r^2 h_1^2}\right]\overline{q}_j = \overline{f}_j, \qquad j = 0, 1, \dots, n-1. \tag{10}$$

For the functions $\overline{q}_j$, boundary conditions may be imposed taking into account the singularities of the differential operator (10) at $r = 0$. Thus, for the Dirichlet and Neumann boundary-value problems under consideration in the circle, when solving them by the method of grids, one should set:

$$\text{Problem D:} \quad \begin{array}{ll} \text{for } r = 0 & \overline{q}_j(0) \text{ is bounded,} \\ \text{for } r = R & \overline{q}_j(R) = \overline{p}_j. \end{array} \tag{11}$$

$$\text{Problem N:} \quad \begin{array}{ll} \text{for } r = 0 & \dfrac{\partial \overline{q}_j}{\partial r} \text{ is bounded,} \\ \text{for } r = R & \dfrac{\partial \overline{q}_j}{\partial r} = \overline{s}_j. \end{array} \tag{12}$$

In turn, equation (10) can be represented, using the nodes* $r = r_k = R\dfrac{2k+1}{2m}$, $k = 0, 1, \dots, m-1$, on the interval $[0, R]$, in the finite-difference form

$$a_k \overline{q}_{j,k+1} - b_k \overline{q}_{j,k} + c_k \overline{q}_{j,k-1} = \overline{f}_{j,k}, \qquad k = 0, 1, \dots, m-1, \tag{13}$$

where $\overline{q}_{j,k} = \overline{q}_j(r_k)$, and the quantities $a_k$, $b_k$, and $c_k$ are determined in the finite-difference approximation of the operator in (10) and in the use of the boundary conditions (11) and (12). The reduction of equation (10) to the form (13) and the solution of the latter by the sweep method have been studied in detail in (3). As a result of solving equation (13), for each index the vector $\overline{Q}(r_k)$ becomes known. The desired vector $Q(r_k)$ is determined from the equality $\overline{Q}(r_k) = H\widetilde{Q}(r_k)$ by multiplying it on the left by the matrix $H$, so that $\widetilde{Q}(r_k) = H\overline{Q}(r_k)$. The values of the components of the vector-matrix $\widetilde{Q}(r_k)$ determine, at each point $(r_k, \varphi_j)$, the function $q_{j,k} = Q(r_k, \varphi_j)$, which is the desired solution of the boundary-value problem in the circle on the polar grid of points. Only the value of the desired function at the origin remains undetermined. But

* The indicated choice of nodal points $r_k$ is characteristic for the Neumann boundary-value problem.

For the Dirichlet boundary-value problem the nodal points $r_k$ are chosen on the basis of the formula:

$$r_k = R\frac{2k+1}{2m+1}, \qquad k = 0, 1, \dots, m.$$

Since the nearest grid node at which the sought function is defined is located at a distance $\varepsilon = r_0$ from the origin, it is evident that, in order to extend the solution to the origin, it is necessary to solve the Dirichlet problem for the original equation (1) in a circle of radius $\varepsilon$. Since $\varepsilon$ is sufficiently small, one can construct approximate formulas of varying accuracy for extending the value $Q(x,y)_{\substack{x=0\\y=0}}$, which will depend on $\varepsilon, K, F(x,y)_{\substack{x=0\\y=0}}$, and $Q(r_0, \varphi_k)$, $k = 0, 1, \dots$. Such formulas can be constructed in accordance with the conditions of the physical problem being solved in each specific case.

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*

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