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Abstract

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MATHEMATICS

M. SHIRINBEKOV

ON RUNGE DOMAINS IN THE SPACE OF SEVERAL COMPLEX VARIABLES

(Presented by Academician N. N. Bogolyubov, 13 II 1962)

1. In the theory of functions of one complex variable the following Runge theorem holds ⁽¹⁾:

Let D be an arbitrary simply connected domain. Then for every function $f(z)$ holomorphic in D there exists a sequence of polynomials $P_n(z)$ such that

$$f(z) = \lim_{n \rightarrow \infty} P_n(z)$$

uniformly inside D , i.e., on every compact set contained in D .

In the space of functions of several complex variables C^n , the theorem just formulated is, generally speaking, false ^(2,3). Therefore, in each particular case one has to establish a theorem on the possibility of uniformly approximating holomorphic functions in a given domain by a sequence of polynomials. Domains for which such an approximation is possible are called Runge domains. Thus, in ⁽⁴⁾ it is established that every convex domain in C^n and every tubular domain are Runge domains.

2. We recall several definitions and facts from the theory of functions of several complex variables.

Let K be a set of functions holomorphic in a domain D . K is called a class ^(3, p. 242) if, together with a function $f(z)$, K contains: a) the derivatives of $f(z)$ of all orders; b) all functions of the form $A[f(z)]^p$, where A is an arbitrary complex number and p is an arbitrary positive integer.

In accordance with this definition, the totality of all functions holomorphic in D , or the totality of all polynomials, forms a class.

Let K be a class of functions holomorphic in a domain D ; let Δ be the intersection of all domains of holomorphy of the functions of the class K . The domain D is called convex with respect to the class K ^(3, p. 246), or, briefly, K -convex, if: a) D is a subdomain of Δ ; b) for every subdomain D_0 of the domain D

($D_0 \Subset D$, the minimal boundary distance of D_0 in D is equal to r) and every finite point $z_0 \in D$ with boundary distance in D less than r , there is in the class K at least one function $f(r)$ such that

$$|f(z_0)| > \max_{z \in D_0} |f(z)|.$$

If K is the class of polynomials, then D is called convex with respect to the class of polynomials. According to Weil's theorem (³, p. 331), a domain D is a Runge domain if and only if its envelope of holomorphy $H(D)$ is convex with respect to some class of polynomials.

3. In the present note we shall prove the following theorems:

Theorem 1. *A domain D is a Runge domain if and only if in D there exists a class K of holomorphic functions satisfying the conditions: 1) the envelope of holomorphy $H(D)$ is K -convex; 2) every function of the class K is uniformly approximable inside D by polynomials.*

Theorem 2. *A semi-tubular domain of the form*

$$T = \{(z, w) : z \in B, \quad V_1(z) < u < V_2(z), \quad |v| < \infty\},$$

where $w = u + iv$; $z = (z_1, z_2, \dots, z_n)$; $V_1(z), -V_2(z)$ are upper semicontinuous in B ; B is a domain of holomorphy in C^n ,

if and only if it is a Runge domain when B is a Runge domain.

4. Proof of Theorem 1. Necessity. Let D be a Runge domain. Then, as the class K , one may choose the class of all functions holomorphic in D . Indeed, first, $H(D)$, as a domain of holomorphy, will be convex with respect to this class. Second, since every function holomorphic in D is uniformly approximated by polynomials, the same also holds in the envelope of holomorphy $H(D)$, for every function holomorphic in D assumes in $H(D)$ the same values as in D .

Sufficiency. Suppose that in the domain D there exists such a class K which satisfies conditions 1) and 2). Condition 1) means that for every subdomain $D_0 \Subset H(D)$ with minimal boundary distance of D_0 in $H(D)$ equal to $r > 0$, and any finite point $z_0 \in H(D)$ with boundary distance less than r , there is a function $f(z) \in K$ such that

$$|f(z_0)| > \max_{z \in D_0} |f(z)|.$$

Let $\varepsilon = |f(z_0)| - \max_{z \in D_0} |f(z)| > 0$. Take a compact set Q which contains the point z_0 and D_0 and is contained in $H(D)$. By condition 2), there exists a polynomial $p(z)$ such that

$$|f(z) - p(z)| < \alpha\varepsilon, \quad z \in Q,$$

where the number $\alpha > 0$ will be chosen below.

Indeed, it is known ([3], p. 274) that if $D_\nu \Subset D_{\nu+1} \Subset D$ and $\lim_{\nu \rightarrow \infty} D_\nu = D$, then $\lim_{\nu \rightarrow \infty} H(D_\nu) = H(D)$.

Therefore to any compact set $Q \subset H(D)$ there corresponds a domain $D' \Subset D$ such that $Q \subset H(D') \subset H(D)$.

By condition 2), there exists a polynomial $p(z)$ such that

$$|f(z) - p(z)| < \alpha\varepsilon, \quad z \in D'. \quad (1)$$

Since every function holomorphic in D assumes in the envelope of holomorphy $H(D)$ the same values as in D , (1) also holds for the compact set Q . Then

$$\begin{aligned} |p(z_0)| &= |p(z_0) - f(z_0) + f(z_0)| \geq |f(z_0)| - |f(z_0) - p(z_0)| \geq \\ &\geq |f(z_0)| - \alpha\varepsilon = \max_{z \in D_0} |f(z)| + \varepsilon(1 - \alpha). \end{aligned} \quad (2)$$

Further,

$$\max_{z \in D_0} |f(z)| \geq \max_{z \in D_0} |p(z)| - |f(z) - p(z)| \geq \max_{z \in D_0} |p(z)| - \alpha\varepsilon. \quad (3)$$

Thus, from (2) and (3) we obtain

$$|p(z_0)| \geq \max_{z \in D_0} |p(z)| + \varepsilon(1 - 2\alpha).$$

Choose $\alpha < 1/2$. Then, obviously,

$$|p(z_0)| > \max_{z \in D_0} |p(z)|,$$

and, consequently, $H(D)$ is convex with respect to the class of polynomials. Therefore, by Weil's theorem it follows that D is a Runge domain. Theorem 1 is proved.

5. To prove Theorem 2, we first formulate a number of lemmas.

Lemma 1. A semitubular domain T is a domain of holomorphy if and only if $V_1(z)$ and $-V_2(z)$ are plurisubharmonic functions.

Lemma 2. The envelope of holomorphy $H(T)$ of the semitube domain T coincides with the semitube domain

$$\tilde{T} = (z, w) : z \in B; \tilde{V}_1(z) < u < \tilde{V}_2(z), \quad |v| < \infty,$$

where $\tilde{V}_1(z), \tilde{V}_2(z)$ are the greatest plurisubharmonic majorants of the functions $V_1(z), -V_2(z)$, respectively.

For the proof of these lemmas in the case $n = 1$, see ⁽⁵⁾.

Lemma 3. Every function f , holomorphic in the semitube domain T and satisfying the condition

$$f(z, w) = f(z, w + 2\pi i),$$

can be expanded in the series

$$f(z, w) = \sum_{-\infty}^{\infty} \varphi_k(z) e^{kw}, \quad (4)$$

which converges uniformly inside T . The coefficients $\varphi_k(z)$ are determined by the formula

$$\varphi_k(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z, \zeta) e^{-k\zeta} dv.$$

It follows from this that semitube domains T may be regarded as domains of uniform convergence of the series (4).

Lemma 4. If $V_1(z), -V_2(z)$ are plurisubharmonic functions in the domain B , then the semitube domain T is the domain of holomorphy of some function $f(z, w)$ satisfying the condition

$$f(z, w) = f(z, w + 2\pi i).$$

Proof. Consider the Hartogs domain

$$G = (z, w) : z \in B, \quad e^{V_1(z)} < |w| < e^{V_2(z)}.$$

Since B is a domain of holomorphy, and $V_1(z), -V_2(z)$ are plurisubharmonic, one can prove that G will be a domain of holomorphy. Therefore there exists a function $f(z, w)$, holomorphic in G and not holomorphic in any larger domain. It follows that the function $f(z, e^w)$ is holomorphic in the semitube domain T and not holomorphic in any larger domain. Lemma 4 is proved.

6. Proof of Theorem 2. The necessity of the condition of the theorem is obvious. Let us prove sufficiency. For this it is enough to show that in the domain T there exists a certain class K of holomorphic functions satisfying the conditions of Theorem 1. We assert that as the indicated class K one may take the totality of all functions $f(z, w)$, holomorphic in the domain T and having periods in the variable w equal to $2\pi i$.

Indeed, it follows from Lemma 4 that the envelope $H(T)$, as a domain of holomorphy of functions of the class K , is K -convex ^(3, p. 248). Further, since, by assumption, B is a Runge domain, the functions $\varphi_k(z)$ in the expansion (4) are

uniformly approximated by polynomials inside B . Then it follows from Lemma 3 that every function $f(z, w)$ of the class K is uniformly approximated by polynomials. Thus conditions 1) and 2) of Theorem 1 are satisfied. Theorem 2 is proved.

From Theorem 2, in the case of two complex variables z, w , we obtain:

Corollary. A semitube domain T is then a Runge domain when the domain B is simply connected.

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