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Abstract

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ASYMPTOTIC BEHAVIOR OF FUNDAMENTAL SOLUTIONS OF HYPOELLIPTIC EQUATIONS WITH TWO VARIABLES AND A PROBLEM WITH CONDITIONS AT INFINITY

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1. Introduction. In this paper we consider equations

$$P\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial y}\right)u = f, \tag{1}$$

where $P(i\partial/\partial x,\ i\partial/\partial y)$ is a hypoelliptic operator [1] in two variables with constant coefficients, acting on functions defined in the whole plane (x, y).

By W we shall denote a class of functions given on the entire plane (x, y), in which there exists a unique solution of equation (1). Usually the classes W are easily found after the asymptotics at infinity of the fundamental solutions of equation (1) have been obtained.

It is known that if equation (1) is hypoelliptic and the characteristic polynomial $P(s, z)$ has no real zeros, then equation (1) has an exponentially decreasing fundamental solution. In Palamodov' s paper [2] the class W was found for any equation (1) whose characteristic polynomial has no real zeros. The asymptotics at infinity of fundamental solutions and the class W for certain special types of equations with a characteristic polynomial having real zeros were obtained by Sommerfeld [3], I. N. Vekua [4], B. P. Paneah [5], and B. R. Vainberg [6].

We shall find the asymptotics of fundamental solutions and the classes W for any hypoelliptic equation in two variables satisfying the following conditions:

1) $P(s, z)$ has constant real coefficients;

2) $P(s, z)$ has real zeros;

3) $\operatorname{grad} P(s, z) \neq 0$ at the real zeros of $P(s, z)$.

Lemma 1. *If the polynomial $P(s, z)$ satisfies conditions 1)-3) and has no multiple factors, then for all values of ψ, $0 \leq \psi < 2\pi$, except for a finite number,* $\operatorname{grad} P(s, z) \neq 0$ *at those roots of the polynomial $P(s, z)$ for which $z_\psi = \sin\psi \cdot s + \cos\psi \cdot z$ is real.*

It follows from Lemma 1 that if $P(s, z)$ satisfies conditions 1)-3) and has no multiple factors, then without loss of generality one may assume that $\operatorname{grad} P(s, z) \neq 0$ at the roots of $P(s, z)$ with real z.

2. Construction of fundamental solutions. From the conditions listed above it follows that the real zeros of the polynomial $P(s, z)$ form several closed smooth curves. Choosing on them, in an arbitrary way, a finite number of points, we divide these curves into arcs. We represent the set of real zeros of $P(s, z)$ as the sum of two sets Λ_1 and Λ_2, where Λ_1 is an arbitrary finite collection of the obtained arcs of the curves $P(s, z) = 0$. If a real root of the polynomial $P(s, z)$ is the boundary of two arcs at once that belong to Λ_j, then we also assign it to the corresponding Λ_j, $j = 1, 2$. The remaining endpoints of the arcs of the curves $P(s, z) = 0$ may be assigned arbitrarily either to Λ_1 or to Λ_2.

Let $s = \xi + i\eta,\ z = \sigma + i\tau$. Consider the three-dimensional space (ξ, σ, η). For each constant $\sigma = c$, $P(s, \sigma)$ vanishes at a finite number of points. For each c we draw a line l_c coinciding with the straight line $\sigma = c,\ \eta = 0$, if on it there are no points where $P(s, \sigma) = 0$. If, however, on this straight line there are points at which $P(s, \sigma) = 0$, then we go around them, remaining in the plane $\sigma = c$, and in such a way that the points of the set Λ_1 remain above (in the direction $\eta > 0$) our line l_c, while the points of the set Λ_2 remain below our line. We denote the obtained set $\{l_c\}$ by H. Consider the integral

$$E(x, y) = \frac{1}{(2\pi)^2} \iint_H \frac{e^{-ixs-iyz}}{P(s, z)}\, dH. \tag{2}$$

Theorem 1. *If the polynomial $P(s, z)$ is hypoelliptic, satisfies conditions 1)—3), and has no multiple factors, then the integral (2) exists and gives a fundamental solution of equation (1).*

If $P(s, z)$ has factors $Q_j^{\lambda_j}(s, z)$, $\lambda_j > 1$, then, by condition 3), the polynomials $Q_j(s, z)$ have no real zeros, and, consequently, as was already noted, the operator

$$\prod_j Q_j^{\lambda_j}\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial y}\right)$$

has an exponentially decreasing fundamental solution. Then the fundamental solution for equation (1) is obtained as the convolution of the exponentially decreasing fundamental solution for the operator

$$\prod_j Q_j^{\lambda_j}\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial y}\right)$$

and the fundamental solution, constructed in the manner described above, for the remaining operator

$$\frac{P}{\prod_j Q_j^{\lambda_j}}\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial y}\right).$$

3. Asymptotics of fundamental solutions.

The integral (2) with respect to the variable s is computed with the aid of residue theory, and the remaining one-dimensional integrals are studied by the method of steepest descent (7).

By $l(\varphi)$ we denote the vector with coordinates $(\sin\varphi, -\cos\varphi)$, where $r\cos\varphi = x$, $r\sin\varphi = y$. By M_1 we denote the totality of boundary points of the sets Λ_j, $j = 1, 2$; by M_2, the set of real roots of the polynomial $P(\xi, \sigma)$ at which the curvature of the lines $P(\xi, \sigma) = 0$ is equal to zero. The sets M_1 and M_2 consist of a finite number of points.

We shall call the following values of φ irregular:

1) all values φ, $\varphi \neq \pi/2$, $\varphi \neq 3\pi/2$, for which $(\operatorname{grad} P(\xi, \sigma), l(\varphi)) = 0$ for at least one point (ξ, σ) from M_1;
2) $\varphi = \pi/2$, $\varphi = 3\pi/2$ in those cases when at least one of the real roots of the polynomial, at which $P'_\xi(\xi, \sigma) = 0$, does not belong to M_1;
3) all values φ for which $(\operatorname{grad} P(\xi, \sigma), l(\varphi)) = 0$ for at least one point (ξ, σ) from M_2.

Irregular directions satisfying the last condition we shall sometimes denote by $\bar{\varphi}$.

There are only finitely many irregular directions.

By $(\xi_n^\varphi, \sigma_n^\varphi)$ we denote all real solutions of the system

$$P'_\xi(\xi, \sigma)\sin\varphi - P'_\sigma(\xi, \sigma)\cos\varphi = 0,$$

$$P(\xi, \sigma) = 0. \tag{3}$$

Within any pair of vertical angles between two neighboring values of φ, the number of solutions of the system (3) will be constant, and all these solutions will be distinct. We number $(\xi_n^\varphi, \sigma_n^\varphi)$ arbitrarily.

Let K_n^φ be the curvature of the curve $P(\xi, \sigma) = 0$ at the point $(\xi_n^\varphi, \sigma_n^\varphi)$; α the greatest order of contact of the curve $P(\xi, \sigma) = 0$ with its tangent; $\delta_1 = \operatorname{sign} x$, $\delta_2 = \operatorname{sign} P_\xi(\xi_n^\varphi, \sigma_n^\varphi)$, and $\delta_3 = \pm 1$, according as the curve $P(\xi, \sigma) = 0$ is convex or concave at the point $(\xi_n^\varphi, \sigma_n^\varphi)$ with respect to the ξ-axis.

Theorem 2. If $P\left(i\dfrac{\partial}{\partial x}, i\dfrac{\partial}{\partial y}\right)$ is a hypoelliptic operator satisfying conditions 1) –3), then the fundamental solution constructed above has the following asymptotics:

$$\frac{\partial^{p+l} E(x,y)}{\partial x^p \partial y^l} = \sum_n \left\{ \frac{\delta_1 \delta_2 i \exp\left(\delta_1 \delta_3 i \frac{\pi}{4}\right)}{\sqrt{2\pi r}\sqrt{K_n^\varphi}\,|\operatorname{grad} P(\xi_n^\varphi, \sigma_n^\varphi)|} [-i\xi_n^\varphi]^p [-i\sigma_n^\varphi]^l \right.$$
$$\left. \times \exp[-i(\xi_n^\varphi \cos\varphi + \sigma_n^\varphi \sin\varphi) r] \right\} + E_{p,l}(x,y),$$

where the summation is carried out, for $x < 0$, over all n for which the point $(\xi_n^\varphi, \sigma_n^\varphi)$ belongs to Λ_1, and for $x > 0$ over all n for which $(\xi_n^\varphi, \sigma_n^\varphi) \in \Lambda_2$. Moreover, for $E_{p,l}(x,y)$, when $r > 1$, the estimate

$$|E_{p,l}(x,y)| < C_{p,l} r^{-\lambda} \prod_{i=1}^{s} |\varphi - \varphi_i|^{\frac{\alpha\lambda - 1}{\alpha - 1}}, \tag{4}$$

holds, where φ_i, $i = 1, 2, \dots, s$, are all the irregular directions, and λ is any number in the interval $1/2 \le \lambda \le 1$.

Remark 1. The following estimate holds:

$$\left| \frac{\delta_1 \delta_2 i \exp\left(\delta_1 \delta_3 i \frac{\pi}{4}\right)}{\sqrt{2\pi}\sqrt{K_n^\varphi}\,|\operatorname{grad} P(\xi_n^\varphi, \sigma_n^\varphi)|} \right| < C \sum_{i=1}^{s} |\varphi - \varphi_i|^{-\frac{1}{2} + \frac{1}{2(\alpha-1)}}.$$

Remark 2. If in some neighborhoods of the irregular directions φ_{i_k}, $k = 1, 2, \dots, t$, $t \le s$, the estimate

$$|\xi_n^\varphi|^p |\sigma_n^\varphi|^l < C_{p,l} \prod_{k=1}^{t} |\varphi - \varphi_{i_k}|^{q_k(p,l)}$$

holds, then instead of (4) $E_{p,l}(x,y)$ may be estimated as follows:

$$|E_{p,l}(x,y)| < C_{p,l} r^{-\lambda} \prod_{i=1}^{s} |\varphi - \varphi_i|^{-\frac{\alpha\lambda - 1}{\alpha - 1}} \prod_{k=1}^{t} |\varphi - \varphi_{i_k}|^{q_k(p,l)}.$$

Remark 3. It is clear that any finite linear combination of the fundamental solutions constructed by us, with coefficients whose sum is equal to one, will again give a fundamental solution, the asymptotics of which will be a linear combination, with the same coefficients, of the formulas already obtained by us.

4. Classes W. Theorem 2 allows us to obtain, for the equations under consideration, classes W. In the plane (x, y) draw the rays: $\varphi = \varphi_\nu$, $0 \le \varphi_\nu < 2\pi$, $\nu = 1, 2, \dots, 2k$. As the φ_ν we take all values of φ, and also $\varphi = \psi_0$ and $\varphi = \psi_0 + \pi$, where ψ_0 is one of the values of ψ, $0 \le \psi < \pi$, for which $\operatorname{grad} P(s, z) \neq 0$ at the roots $P(s, z)$ with real z. The remaining φ_ν are chosen arbitrarily, taking care only that together with each ray $\varphi = \varphi_\nu$ there is drawn the ray $\varphi = \varphi_{\nu'}$, where $\varphi_{\nu'}$ is the direction opposite to φ_ν. For convenience we number the rays drawn so that $\varphi_{\nu+1} > \varphi_\nu$, $\nu = 1, 2, \dots, 2k-1$. Sometimes we shall denote φ_1 by φ_{2k+1}. Let $\psi_0 = \varphi_i$, $\psi_0 + \pi = \varphi_{k+i}$. Denote by m_ν, $\nu = 1, 2, \dots, 2k$, the number of real solutions of system (3) for φ satisfying the inequalities:

$$\tan \varphi_\nu < \tan \varphi < \tan \varphi_{\nu+1}.$$

We shall number the solutions of the system (3), for convenience, in such a way that, for $\operatorname{tg} \varphi' < \operatorname{tg} \varphi < \operatorname{tg} \varphi''$, the functions ξ_n^φ and σ_n^φ are, for each n, continuous functions of φ. [Here φ' and φ'' are any two neighboring values of φ, chosen from the φ' s equal to $\bar{\varphi}$, ψ_0, and $\psi_0 + \pi$.] Let us take arbitrarily $\delta_{n\nu}$, $n = 1, 2, \dots, m_\nu$, $\nu = 1, 2, \dots, k$, equal either to 1 or to 0, and let $\delta_{n,k+\nu} = 1 - \delta_{n\nu}$.

Theorem 3. *If $P\left(i\dfrac{\partial}{\partial x}, i\dfrac{\partial}{\partial y}\right)$ is a hypoelliptic operator satisfying conditions 1)–3), and $f(x, y)$ is any finite summable function,* then there exists a unique solution of equation (1) in the following class of functions W: the function $u \in W$, if for $\varphi_\nu < \varphi < \varphi_{\nu+1}$, $\nu = 1, 2, \dots, 2k$, $u(x, y)$ is representable in the form: for those ν for which $0 < \sum_n \delta_{n\nu} < m_\nu$, $u = \sum_n \delta_{n\nu} u_{n\nu}$; for those ν for which $\sum_n \delta_{n\nu} = 0$, $u = v_\nu$; for those ν for which $\sum_n \delta_{n\nu} = m_\nu$, $u = g_\nu$, where $u_{n\nu}(x, y)$, $v_\nu(x, y)$, $g_\nu(x, y)$ are arbitrary functions, defined for $\varphi_\nu < \varphi < \varphi_{\nu+1}$, and, in a neighborhood of infinity, for some λ from the interval

$$\frac{1}{2} < \lambda < \frac{1}{2} + \frac{1}{\alpha} + p + l,$$

of order lower than the order of equation (1), satisfying the estimates:

$$|u_{n\nu}(x, y)| < C r^{-1/2} \left(|\varphi - \varphi_\nu|\,|\varphi - \varphi_{\nu+1}|\right)^{-\frac{1}{2} + \frac{1}{2(\alpha - 1)}},$$

$$\left|[-i\xi_n^\varphi]^{-p} [-i\sigma_n^\varphi]^{-l} \frac{\partial^{p+l} u_{n\nu}(x, y)}{\partial x^p \partial y^l} - u_{n\nu}(x, y)\right| <$$

$$< C r^{-\lambda} \left(|\varphi - \varphi_\nu|\,|\varphi - \varphi_{\nu+1}|\right)^{-\frac{\alpha\lambda - 1}{\alpha - 1}}, \tag{5}$$

$$|g_\nu(x, y)| < C r^{-1/2} \left(|\varphi - \varphi_\nu|\,|\varphi - \varphi_{\nu+1}|\right)^{-\frac{1}{2} + \frac{1}{2(\alpha - 1)}},$$

$$|v_\nu(x, y)| < C r^{-\lambda} \left(|\varphi - \varphi_\nu|\,|\varphi - \varphi_{\nu+1}|\right)^{-\frac{\alpha\lambda - 1}{\alpha - 1}}.$$

If $\varphi_i \neq \bar{\varphi}$ and in a neighborhood of φ_i the conditions imposed on $u(x, y)$ vary continuously, then in the right-hand sides of the estimates (5) one must, if they occur, omit the factors $|\varphi - \varphi_i|$ and $|\varphi - \varphi_{i+1}|$.

Remark 1. In Theorem 3, the classes W corresponding to the fundamental solutions constructed in § 2 have been obtained. In an analogous way one can obtain the classes W corresponding to the fundamental solutions mentioned in Remark 3 to Theorem 2.

Remark 2. Theorems 1, 2, 3 remain valid if the polynomial $P(s, z)$ has factors with complex coefficients without real zeros.

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* The function $f(x, y)$ could have been taken from a broader class of functions, but this question is not of interest to us at present.

Note: Figure translations are in progress. See original paper for figures.

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