



Soviet-era science, translated into English

V. B. KOROTKOV

1962

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196201.81850>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

V. B. KOROTKOV

ABSTRACT SET FUNCTIONS AND EMBEDDING THEOREMS

(Presented by Academician S. L. Sobolev on 31 III 1962)

Let Ω be a simply connected domain of n -dimensional Euclidean space R_n . Denote by $\Sigma(\Omega)$ the collection of all subsets of Ω having finite Lebesgue measure. The collection of all abstract additive set functions defined on $\Sigma(\Omega)$ and taking values in a (B) -space X forms a vector manifold $\Phi(X, \Omega)$.

We shall say that $\Phi(E)$ from $\Phi(X, \Omega)$ belongs to $\Phi_1(X, \Omega)$ ⁽¹⁾ (respectively, to $\Phi_p(X, \Omega)$, $1 < p < \infty$ ⁽¹⁾), if the inequality (1) (respectively, the inequality (2)) holds:

$$\|\Phi(E)\|_{\Phi_1(X, \Omega)} = \sum_{\substack{E_1 \in \Sigma(\Omega), E_2 \in \Sigma(\Omega) \\ E_1 \cap E_2 = \emptyset}} \|\Phi(E_1) - \Phi(E_2)\|_X < \infty; \quad (1)$$

$$\|\Phi(E)\|_{\Phi_p(X, \Omega)} = \sup_{\omega} \frac{\left\| \int_{\Omega} \omega(x) d_x \Phi(E) \right\|_X}{\|\omega(x)\|_{L_{p'}(\Omega)}} < \infty, \quad (2)$$

where \emptyset is the empty set, $1 < p < \infty$, $1/p + 1/p' = 1$, and the least upper bound is taken over all finite linear combinations of characteristic functions of sets from $\Sigma(\Omega)$. We shall call such functions step functions. Note that

$$\sup_{E \in \Sigma(\Omega)} \|\Phi(E)\|_X \leq \|\Phi(E)\|_{\Phi_1(X, \Omega)} \leq 2 \sup_{E \in \Sigma(\Omega)} \|\Phi(E)\|_X. \quad (3)$$

We shall call a function $\Phi(E) \in \Phi(X, \Omega)$: 1) countably additive, if

$$\lim_{N \rightarrow \infty} \left\| \Phi \left(\bigcup_{i=1}^{\infty} E_i \right) - \sum_{i=1}^N \Phi(E_i) \right\|_X = 0$$

for every sequence $\{E_i\}$ of pairwise disjoint sets from $\Sigma(\Omega)$ such that $\bigcup_{i=1}^{\infty} E_i \subset \Sigma(\Omega)$; 2) absolutely continuous in the norm $\|\cdot\|_{\Phi_p}$, $1 \leq p < \infty$, if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|\Phi(E)\|_{\Phi_p(X, \Lambda)} < \varepsilon$, if $m\Lambda < \delta$; 3) continuous under translation in the norm $\|\cdot\|_{\Phi_p}$, $1 \leq p < \infty$, if

$$\lim_{|\vec{h}| \rightarrow 0} \|\Phi(E + \vec{h}) - \Phi(E)\|_{\Phi_p(X, \Omega)} = 0,$$

where $\Phi(E + \vec{h}) = \Phi((E + \vec{h}) \cap \Omega)$. If the function Φ is countably additive, then $\|\Phi\|_{\Phi_1(X, \Omega)} < \infty$ ⁽²⁾, p. 319).

Consider the following spaces: $\tilde{\Phi}_1(X, \Omega)$ ⁽³⁾ is the space of all countably additive functions with norm (1); $\tilde{\Phi}_p(X, \Omega)$, $1 < p < \infty$ ⁽¹⁾, is the space of all functions absolutely continuous in the norm $\|\cdot\|_{\Phi_p}$, with norm $\|\cdot\|_{\Phi_p}$; $\Psi_1(X, \Omega)$ ⁽¹⁾ is the space of all functions from $\tilde{\Phi}_1(X, \Omega)$ continuous under translation in the norm (1); $\Psi_p(X, \Omega)$, $1 < p < \infty$ ⁽¹⁾, is the space of all functions from $\tilde{\Phi}_p(X, \Omega)$ continuous under translation in the norm (2).

All these spaces are Banach spaces, and, in the case of a bounded domain Ω , the following embeddings hold:

$$\Phi_1(X, \Omega) \leftarrow \tilde{\Phi}_1(X, \Omega) \leftarrow \tilde{\tilde{\Phi}}_1(X, \Omega) \leftarrow \Psi_1(X, \Omega); \quad (4)$$

$$\tilde{\Phi}_1(X, \Omega) \leftarrow \Phi_p(X, \Omega) \leftarrow \tilde{\tilde{\Phi}}_p(X, \Omega) \leftarrow \Psi_p(X, \Omega). \quad (5)$$

The embeddings (4) are easily verified with the aid of Pettis' theorem (see, for example, ⁽²⁾, p. 318). The embedding $\Phi_p(X, \Omega) \rightarrow \tilde{\tilde{\Phi}}_1(X, \Omega)$ follows from (3) and the following estimate (1):

$$\|\Phi(I)\|_X = (mI)^{1/p'} \frac{\|\Phi(I)\|_X}{(mI)^{1/p'}} = (mI)^{1/p'} \frac{\left\| \int_{\Omega} \chi_I(x) d_x \Phi(E) \right\|_X}{\|\chi_I(x)\|_{L_{p'}(\Omega)}} \leq (mI)^{1/p'} \|\Phi\|_{\Phi_p(X, \Omega)}. \quad (6)$$

The embedding $\Psi_p(X, \Omega) \rightarrow \tilde{\tilde{\Phi}}_p(X, \Omega)$ was proved in ⁽¹⁾.

Let $\Phi(E) \in \Phi(Y^*, \Omega)$, where Y^* is the space conjugate to the B -space Y . We shall call $\Phi \in \Phi(Y^*, \Omega)$ $(*)$ -weakly absolutely continuous (and denote the totality of all such functions by $\tilde{\Phi}_{1^*}(Y^*, \Omega)$) if, for every $g \in Y$, the set function $\Phi(E)g$ is absolutely continuous. It can be shown that if $\Phi(E) \in \tilde{\Phi}_{1^*}(Y^*, \Omega)$, then $\|\Phi\|_{\Phi_1(Y^*, \Omega)} < \infty$, and $\tilde{\Phi}_{1^*}(Y^*, \Omega)$ is a B -space with norm (1).

Let $\Phi(E) \in \tilde{\Phi}_{1^*}(Y^*, \Omega)$. Then

$$\Phi(E)g = \int_E \varphi_g(x) dx, \quad \varphi_g(x) = \frac{d}{dx}[\Phi(E)g], \quad g \in Y, \quad (7)$$

where $\frac{d}{dx}[\Phi(E)g]$ is the Radon-Nikodym derivative ⁽⁴⁾.

Lemma 1. 1) Let $\Phi(E) \in \tilde{\Phi}_{1^*}(Y^*, \Omega)$; then

$$\|\Phi\|_{\Phi_{1^*}(Y^*, \Omega)} = \sup_{\|g\|_Y \leq 1} \|\varphi_g\|_{L_1(\Omega)}.$$

2) Let $1 < p < \infty$, $\Phi(E) \in \Phi_p(Y^*, \Omega)$; then

$$\|\Phi\|_{\Phi_p(Y^*, \Omega)} = \sup_{\|g\|_Y \leq 1} \|\varphi_g\|_{L_p(\Omega)}.$$

Let B_1 and B_2 be (B) -spaces. By $(B_1 \rightarrow B_2)$ we denote the space of all linear continuous operators acting from B_1 into B_2 . The set of linear operators from $(B_1 \rightarrow B_2)$ possessing some property N will be denoted by $(B_1 \rightarrow B_2, N)$. An operator $T \in (B_1 \rightarrow B_2)$ will be called an isomorphism if $TB_1 = B_2$ and T maps B_1 onto B_2 one-to-one.

Theorem 1. 1) $\tilde{\Phi}_{1^*}(Y^*, \Omega) \leftrightarrow (Y \rightarrow L_1(\Omega))$; 2) if $m\Omega < \infty$, then $\tilde{\Phi}_1(Y^*, \Omega) \leftrightarrow (Y \rightarrow L_1(\Omega), \text{ weakly completely continuous})$; 3) if $1 < p < \infty$, then $\Phi_p(Y^*, \Omega) \leftrightarrow (Y \rightarrow L_p(\Omega))$; 4) if Ω is a bounded domain and $1 \leq p < \infty$, then $\Psi_p(Y^*, \Omega) \leftrightarrow (Y \rightarrow L_p(\Omega), \text{ completely continuous})$. Here the symbol \leftrightarrow , connecting two spaces, means that these spaces are isometrically isomorphic, and the isometric isomorphism is defined by the equations

$$T_\Phi(g) := \frac{d}{dx} [\Phi(E)g], \quad g \in Y; \quad (8)$$

$$\Phi_T(E)g = \int_E T(g)(x) dx, \quad g \in Y. \quad (9)$$

Indeed, in ⁽⁵⁾ (see also ⁽²⁾, p. 498) it is shown that (8), (9) define an isomorphism between the corresponding spaces of items 1-3. The isometry follows from Lemma 1. We note that earlier only the following estimate was known:

$$\sup_{E \in \Sigma(\Omega)} \|\Phi(E)\|_{Y^*} \leq \|T_\Phi\| \leq 2 \sup_{E \in \Sigma(\Omega)} \|\Phi(E)\|_{Y^*}.$$

4) immediately

follows from the isometry, if one observes that

$$\|\Phi(E + \vec{h}) - \Phi(E)\|_{\Phi_p(Y^*, \Omega)} = \sup_{\|g\|_Y \leq 1} \|T_\Phi(g)(x + \vec{h}) - T_\Phi(g)\|_{L_p(\Omega)}.$$

Theorem 2. The following four assertions are equivalent, if Ω is a bounded domain, $1 \leq p < \infty$: 1) $\Phi(E) \in \Psi_p(X, \Omega)$; 2) $\Phi(E) \in \tilde{\Phi}_p(X, \Omega) \cap \Psi_1(X, \Omega)$; 3)

$\Phi(E) \in \tilde{\Phi}_p(X, \Omega)$ and the set of values of the function Φ is compact in $L_p(\Omega)$;
4) the set

$$M_{p,\Phi} \left\{ \varphi_f(x) = \frac{d}{dx} [f\Phi(E)], \|f\|_X \leq 1 \right\}$$

is compact in $L_p(\Omega)$.

Theorem 3. The set of values of a function Φ from $\tilde{\Phi}_1(X, \Omega)$ is separable.

Definition 1. We shall say that $\Phi(E) \in \Phi_p^{(l)}(X, \Omega)$ if $\Phi(E) \in \tilde{\Phi}_1(X, \Omega)$ and all generalized derivatives* $\Phi^{(\alpha)}(E)$ of order l of the function $\Phi(E)$ belong to $\Phi_p(X, \Omega)$. The norm in $\Phi_p^{(l)}(X, \Omega)$ is defined by the equality

$$\|\Phi(E)\|_{\Phi_p^{(l)}(X, \Omega)} = \|\Phi(E)\|_{\Phi_1(X, \Omega)} + \sum_{|\alpha|=l} \|\Phi^{(\alpha)}(E)\|_{\Phi_p(X, \Omega)}. \quad (10)$$

By definition we put $\Phi_p^{(0)}(X, \Omega) = \Phi_p(X, \Omega)$. From the fact that $\Phi(E) \in \Phi_p^{(l)}(Y^*, \Omega)$ it follows that

$$\Phi(E)g = \int_E \varphi_g(x) dx, \quad \Phi^{(\alpha)}(E)g = \int_E D^{(\alpha)}\varphi_g(x) dx, \quad g \in Y. \quad (11)$$

Definition 2. Let $\lambda \geq 0$, $\lambda = \bar{\lambda} + \alpha$, $\bar{\lambda}$ an integer, $0 < \alpha < 1$. By definition $\Phi(E) \in \Phi_p^{(\lambda)}(Y^*, \Omega)$ if $\Phi(E) \in \Phi_p^{(\bar{\lambda})}(Y^*, \Omega)$ and if the norm is finite (see (11))

$$\begin{aligned} \|\Phi(E)\|_{\Phi_p^{(\lambda)}(Y^*, \Omega)} &= \|\Phi(E)\|_{\Phi_p^{(\bar{\lambda})}(Y^*, \Omega)} + \\ &+ \sum_{|\gamma|=\bar{\lambda}} \sup_{\|g\|_{Y^*} \leq 1} \left(\int_{\Omega} \frac{|D^{(\gamma)}\varphi_g(x) - D^{(\gamma)}\varphi_g(y)|^p}{|x-y|^{n+p\alpha}} dx dy \right)^{1/p}. \end{aligned} \quad (12)$$

If X is an arbitrary (B) -space, then we shall say that $\Phi(E) \in \Phi_p^{(\lambda)}(X, \Omega)$ if $A\Phi(E) \in \Phi_p^{(\lambda)}(X^{**}, \Omega)$, where the operator A is defined by the equation

$$Ax = E, \quad x \in X, \quad E \in X^{**} \text{ and for every } f \in X^* \quad E(f) = f(x). \quad (13)$$

Definition 3. We shall say that $\Phi(E) \in H_p^{(r)}(X, R_n)$; $r = \bar{r} + \alpha$, \bar{r} an integer, $0 < \alpha \leq 1$, if $\Phi(E) \in \Phi_p(X, \Omega)$ and has all unmixed generalized derivatives up to order \bar{r} , belonging to $\Phi_p(X, \Omega)$, and moreover

$$\left\| \Phi_{x_i}^{(\vec{r})}(E + \vec{h}_i) - \Phi_{x_i}^{(\vec{r})}(E) \right\|_{\Phi_p(X, R_n)} < M |\vec{h}_i|^\alpha,$$

if $0 < \alpha < 1$; $i = 1, 2, \dots, n$; (14)

$$\left\| \Phi_{x_i}^{(\vec{r})}(E + \vec{h}_i) - 2\Phi_{x_i}^{(\vec{r})}(E) + \Phi_{x_i}^{(\vec{r})}(E - \vec{h}_i) \right\|_{\Phi_p(X, R_n)} < M |\vec{h}_i|,$$

if $\alpha = 1$; $i = 1, 2, \dots, n$. (15)

* The generalized derivative $\Phi^{(\alpha)}(E)$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, is defined

by the integral identity

$$\int_{\Omega} \frac{\partial^{\alpha} \omega}{\partial x^{\alpha}} d_x \Phi(E) = (-1)^{|\alpha|} \int_{\Omega} \omega d_x \Phi^{(\alpha)}(E), \quad \omega \in C_1^0(\Omega) \quad (1).$$

$H_p^{(r)}(X, R_n)$ becomes a (B) -space if one introduces the norm

$$\|\Phi\|_{H_p^{(r)}(X, R_n)} = \|\Phi\|_{\Phi_p(X, R_n)} + M_{\Phi}, \quad (16)$$

where M_{Φ} is the smallest constant for which inequalities (14), (15), $i = 1, 2, \dots, n$, hold.

Theorem 4. 1) $\Phi_p^{(l)}(Y^*, \Omega) \sim (Y \rightarrow W_p^{(l)}(\Omega))$; 2) $\Phi_p^{(\lambda)}(Y^*, \Omega) \sim (Y \rightarrow W_p^{(\lambda)}\Omega)$; 3) $H_p^{(r)}(Y, \Omega) \sim (Y \rightarrow H_p^{(r)}(R_n))$; here the symbol \sim , connecting two spaces, means that these spaces are isomorphic and the isomorphism is defined by equations (8), (9); moreover

$$\|T_{\Phi}\| \leq \|\Phi\|_{\Phi_p^{(l)}(Y^*, \Omega)} \leq (1 + N_l) \|T_{\Phi}\|,$$

$$\|T_{\Phi}\| = \sup_{\|g\|_{Y \leq 1}} \left\| \frac{d}{dx} [\Phi(E)g] \right\|_{W_p^{(l)}(\Omega)} ;$$

$$\|T_{\Phi}\| \leq \|\Phi\|_{\Phi_p^{(\lambda)}(Y^*, \Omega)} \leq (1 + 2N_{\lambda}) \|T_{\Phi}\|, \quad \|T_{\Phi}\| = \sup_{\|g\|_{Y \leq 1}} \left\| \frac{d}{dx} [\Phi(E)g] \right\|_{W_p^{(\lambda)}(\Omega)} ;$$

$$\|T_{\Phi}\| \leq \|\Phi\|_{H_p^{(r)}(Y^*, R_n)} \leq 2 \|T_{\Phi}\|,$$

$$\|T_\Phi\| = \sup_{\|g\|_Y \leq 1} \left\| \frac{d}{dx} [\Phi(E)g] \right\|_{H_p^{(r)}(R_n)},$$

where N_l is the number of all distinct generalized derivatives of order l .

The theorem establishes the general form of a linear continuous operator acting from a (B) -space into $W_p^{(l)}$, $W_p^{(\lambda)}$, $\lambda \geq 0$, $H_p^{(r)}(R_n)$.

Theorem 5. Let $1 < p_1, p_2 < \infty$. Then: 1) if $W_{p_1}^{(\lambda_1)}(\Omega_{n_1}) \rightarrow W_{p_2}^{(\lambda_2)}(\Omega_{n_2})$, then $\Phi_{p_1}^{(\lambda_1)}(X, \Omega_{n_1}) \rightarrow \Phi_{p_2}^{(\lambda_2)}(X, \Omega_{n_2})$; 2) if $H_{p_1}^{(r_1)}(R_{n_1}) \rightarrow H_{p_2}^{(r_2)}(R_{n_2})$, then $H_{p_1}^{(r_1)}(X, R_{n_1}) \rightarrow H_{p_2}^{(r_2)}(X, R_{n_2})$. Here \rightarrow denotes, as usual, an embedding or an extension (if $\dim \Omega_{n_1} < \dim \Omega_{n_2}$), accompanied by the corresponding inequality for the norms ((6), p. 66). It is assumed that the extension (in the case when $\dim \Omega_{n_1} < \dim \Omega_{n_2}$) is carried out by means of a linear continuous operator V .

Theorem 6. Let $\Phi(E) \in \Phi_p^{(\lambda)}(X, \Omega)$ and $\lambda p > n$. Then

$$\Phi(E) = \int_E \varphi(x) dx, \quad E \in \Sigma(\Omega),$$

where $\varphi(x)$ is a continuous abstract function of the point $x \in \Omega$ with values in X , and

$$\sup_{x \in \Omega} \|\varphi(x)\|_X \leq c \|\Phi\|_{\Phi_p^{(\lambda)}(X, \Omega)},$$

where the constant c does not depend on the function Φ .

Theorem 6 is a strengthening of one theorem of S. L. Sobolev ((1), Theorem 27).

V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
7 III 1962

REFERENCES

1. S. L. Sobolev, *Fund. Math.*, **47**, 3 (1959).
2. N. Dunford, J. Schwartz, *Linear Operators*, 1, N. Y., 1958.
3. Yu. I. Lyuberman, *DAN*, **140**, No. 4 (1961).
4. S. Saks, *Theory of the Integral*, II, 1949.

5. L. V. Kantorovich, B. Z. Vulich, *Comp. Math.*, **5**, 119 (1938).

6. S. M. Nikol'skii, *UMN*, **16**, issue 5 (101) (1961).

7. S. Banach, *Théorie des opérations linéaires*, Warszawa, 1932.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.