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Abstract

Full Text

Mathematics

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On Continuous Additive Functionals of a Multidimensional Wiener Process

(Presented by Academician A. N. Kolmogorov, 23 X 1961)

Theorem. Let $X = (x_t, \infty, \mathcal{M}_t, P_x)$ be a multidimensional Wiener process; φ_t^s, $0 \le s \le t < \infty$, a continuous almost homogeneous additive functional of X (see the definition in ([1])). Then there exist: a) an increasing sequence of closed sets K_n such that x_t leaves, in finite time, only a finite number of these sets (the process is bounded relative to the system of sets $\{K_n\}$); b) a sequence of numerical functions $F_n(x)$ and c) a sequence of vector functions $G_n(x)$ such that, with probability 1, up to the moment of exit from the set K_n,

$$\varphi_t^s = F_n(x_t) - F_n(x_s) + \int_s^t G_n(x_u)\, dx_u.$$

The theorem generalizes to a broad class of diffusion processes and to functionals that are continuous from the right and have a left limit at every point (in the case of a Wiener process all such functionals are continuous). In the case of a one-dimensional Wiener process the results take an especially simple form. Namely, every continuous almost homogeneous additive functional can be written in the form

$$\varphi_t^s = F(x_t) - F(x_s) + \int_s^t G(x_u)\, dx_u,$$

where $F(x)$ is a continuous function; $G(x)$ is a function square-integrable on every finite interval. Conversely, to every pair of such functions there corresponds a functional of the described form.

For simplicity we shall carry out the proof for a three-dimensional Wiener process and for functionals that remain constant after exit from a certain ball (the proof in the general case can be reduced to the consideration of such functionals). Since a three-dimensional Wiener process sooner or later leaves any ball, the functional φ_t^s can naturally be extended to $t = \infty$.

For the proof it is enough to construct a sequence of functionals $\varphi_t^s(n)$, having bounded expectation and variance and such that if, for all $u \le t$, $x_u \in K_n$, then

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$\varphi_t^s(n) = \varphi_t^s(n+1) = \dots = \varphi_t^s$. For the functional $\varphi_t^s(n)$ put $F_n(x) = -M_x \varphi_\infty^0$; then $\varphi_t^s(n) - F_n(x_t) + F_n(x_s)$ is a functional having bounded variance and expectation equal to zero; hence

$$\varphi_t^s(n) - F_n(x_t) + F_n(x_s) = \int_s^t G_n(x_u)\, dx_u$$

(see ([2])).

The main difficulty consists in constructing the functionals $\varphi_t^s(n)$. It is overcome with the aid of an idea due to A. V. Skorokhod ([3]). Put

$$\Phi_\lambda(x) = M_x \exp\left\{-\lambda \sup_{0\le s\le t<\infty} |\varphi_t^s|\right\}.$$

We shall state a lemma belonging to Skorokhod.

Lemma. *If everywhere $\Phi_\lambda(x) \geqslant \varepsilon > 0$, then the functional φ_t^s has bounded expectation and variance.*

Let us outline the proof. Choose N such that $e^{-\lambda N} < \varepsilon$. Denote by τ_1 the first moment when $|\varphi_t^0| = N$ ($\tau_1 = \infty$ if for all t, $|\varphi_t^0| < N$); by τ_2, the first moment after τ_1 when $|\varphi_t^{\tau_1}| = N$, or ∞ if $|\varphi_t^{\tau_1}| < N$, and so on. It is proved that

$$P_x\{\tau_n < \infty\} \leqslant \left(\frac{1-\varepsilon}{1-e^{-\lambda N}}\right)^n. \tag{1}$$

(We note for later that for $n = 1$ the formula is true if $\Phi_\lambda(x) \geqslant \varepsilon$ not for all x, but only for the x appearing in the formula. This is a simple consequence of Chebyshev' s inequality.) Formula (1) proves that the distribution of

$$\sup_{0\leqslant s\leqslant t<\infty} |\varphi_t^s|$$

is majorized by an exponential distribution; therefore this random variable, and still more φ_t^s, has all moments, moreover bounded with respect to x, s, t.

Suppose that $\Phi_\lambda(x) \geqslant \varepsilon$ in some domain A. In this case A. V. Skorokhod [3] showed that one can construct a functional ψ_t^s, equal to φ_t^s up to the exit from some smaller domain and having all moments. The functional ψ_t^s is constructed as follows. Let $f(x)$ be a twice continuously differentiable function, equal to zero outside A and to one in a somewhat smaller domain. According to the change-of-variables formula in the stochastic integral [4],

$$f(x_t) - f(x_s) = \int_s^t g(x_u)\, dx_u + \int_s^t h(x_u)\, du. \tag{2}$$

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(Here $g(x) = \operatorname{grad} f(x)$, $h(x) = \frac{1}{2}\Delta f(x)$.) The functional ψ_t is constructed by the formula

$$\psi_t^s = f(x_t)\varphi_t^s - \int_s^t \varphi_u^s g(x_u)\, dx_u - \int_s^t \varphi_u^s h(x_u)\, du. \tag{3}$$

This gives a functional that remains constant while x_t is outside A.

However, it is not always the case that $\Phi_\lambda(x) \geqslant \varepsilon$ in some domain; the set where this inequality holds may be nowhere dense, and then a smooth function f, equal to zero outside this set, is identically zero. It turns out that here one can use nonsmooth functions for which a formula analogous to (2) holds.

Put $f_i(x) = \min(3 - 3\Phi_\lambda(x), i)$ $(i = 1, 2)$. It is easy to show that $f_i(x)$ is a nonnegative superharmonic function tending to zero at infinity. Since the function f_i is bounded, it is possible (see [5]) to construct a nonnegative functional ${}_i\chi_t^s$ by the formula

$${}_i\chi_t^s = f_i(x_s) - f_i(x_t) - \int_s^t g_i(x_u)\, dx_u,$$

where $g_i(x)$ is the gradient of $f_i(x)$. In addition, $M_x \int_0^\infty g_i^2(x_u)\, du$ and $M_x\, {}_i\chi_\infty^0$ turn out, for all x, to be less than 20. Now put $f(x) = 1 + f_1(x) - f_2(x)$, $g(x) = g_1(x) - g_2(x)$, $\chi_t^s = {}_2\chi_t^s - {}_1\chi_t^s$. There is a formula analogous to (2):

$$f(x_t) - f(x_s) = \int_s^t g(x_u)\, dx_u + \chi_t^s. \tag{4}$$

Now we construct the functional ψ_t^s by the formula

$$\psi_t^s = \varphi_t^s f(x_t) - \int_s^t \varphi_u^s g(x_u)\, dx_u - \int_s^t \varphi_u^s \chi(du). \tag{5}$$

(χ_t^s is the increment of some function of bounded variation; integration with respect to $\chi(du)$ means integration with respect to this function of bounded variation.) Using formula (4) and the properties of the stochastic integral (see (†)), it is easy to prove that ψ_t^s is again a continuous almost homogeneous additive functional. Moreover, up to the exit from the set $\{x : \Phi_\lambda(x) \geqslant 2/3\}$, where $f(x) = 1$ and $g(x) = 0$, $\psi_t^s = \varphi_t^s$, while as long as x_t is in the set $\{x : \Phi_\lambda(x) < 1/3\}$, the functional ψ_t^s remains constant.

Let us estimate

$$\Psi_1(x) = M_x \exp\{-\sup_{0 \leqslant s \leqslant t < \infty} |\psi_t^s|\}$$

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from below. Suppose first that $\Phi_\lambda(x) \geqslant 1/3$. Put $\overset{0}{\varphi}_s = \varphi_s^0$ if $\sup_{0\leqslant u\leqslant s} |\varphi_u^0| < N$, and $\overset{0}{\varphi}_s = 0$ otherwise, and consider the expression

$$\overset{0}{\psi}_t = \overset{0}{\varphi}_t f(x_t) - \int_0^t \overset{0}{\varphi}_s g(x_s)\, dx_s - \int_0^t \overset{0}{\varphi}_s \chi(ds). \tag{6}$$

The first term of this expression does not exceed N. The second term is a martingale, and

$$M_x \left(\int_0^\infty \overset{0}{\varphi}_s g(x_s)\, dx_s \right)^2 \leqslant 80N^2;$$

hence, by Kolmogorov' s inequality,

$$P_x \left\{ \sup_{0\leqslant t\leqslant\infty} \left| \int_0^t \overset{0}{\varphi}_s g(x_s)\, dx_s \right| \geqslant \frac{9N}{\delta} \right\} \leqslant \delta^2.$$

As for the last term, we have

$$M_x \int_0^\infty |\overset{0}{\varphi}_s|\, |\chi(ds)| \leqslant 40N,$$

and

$$P_x \left\{ \sup_{0\leqslant t\leqslant\infty} \left| \int_0^t \overset{0}{\varphi}_s \chi(ds) \right| \geqslant \frac{40N}{\delta} \right\} \leqslant \delta.$$

Thus, for any $\delta > 0$ and $N > 0$ there exists a constant

$$A = N + \frac{49N}{\delta}$$

such that

$$P_x \left\{ \sup_{0\leqslant t<\infty} |\overset{0}{\psi}_t| \geqslant A \right\} \leqslant \delta + \delta^2.$$

Now choose N so that $e^{-\lambda N} < 1/3$, and δ so that

$$1 - \frac{1/3}{1 - e^{-\lambda N}} - \delta - \delta^2 > 0.$$

With probability not less than

$$1 - \frac{1/3}{1 - e^{-\lambda N}}$$

(see formula (1)),

$$\sup_{0\leqslant t\leqslant\infty} |\varphi_t^0| < N;$$

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in this case $\psi_t^s = \overset{0}{\psi}_t - \overset{0}{\psi}_s$. Hence we obtain

$$P_x\left\{\sup_{0\leqslant s\leqslant t\leqslant\infty}|\psi_t^s| \geqslant 2A\right\} \leqslant \frac{1/3}{1-e^{-\lambda N}} + \delta + \delta^2,$$

$$\Psi_1(x) = M_x \exp\{-\sup_{0\leqslant s\leqslant t<\infty}|\psi_t^s|\} \geqslant e^{-2A}\left[1 - \frac{1/3}{1-e^{-\lambda N}} - \delta - \delta^2\right] = \varkappa > 0.$$

We now show that also for $\Phi_\lambda(x) < 1/3$, $\Psi_1(x) \geqslant \varkappa$. Denote by τ the first exit time from the set $\{x : \Phi_\lambda(x) < 1/3\}$. It is easy to prove that with probability 1, $\tau < \infty$. Clearly, for $s, t < \tau$, $\psi_t^s = 0$. Therefore

$$\sup_{0\leqslant s\leqslant t<\infty}|\psi_t^s| = \sup_{\tau\leqslant s\leqslant t<\infty}|\psi_t^s|,$$

whence, using the almost homogeneity of the function—

of the functional ψ_t^s, one can infer that $\Psi_1(x) = M_x\Psi_1(x_\tau)$. Since $\{x : \Phi_\lambda(x) < 1/3\}$ is an open set, x_τ does not belong to it; thus, $\Phi_\lambda(x_\tau) \geq 1/3$ and $\Psi_1(x_\tau) \geq \varkappa$. Therefore $\Psi_1(x)$ is everywhere $\geq \varkappa$.

Now choose a sequence $\lambda_n \to 0$ and take as the set K_n the set $\{x : \Phi_{\lambda_n}(x) \geq 2/3\}$; the functional $\varphi_t^s(n) = \psi_t^s$ coincides with φ_t^s up to the time of exit from K_n and has bounded mathematical expectation and variance. It remains to prove that the process X is bounded with respect to the system of sets $\{K_n\}$. It is easy to prove that the probability that x_t ever leaves K_n does not exceed $3(1-\Phi_{\lambda_n}(x))$. As $n \to \infty$, the probability of this event tends to zero, since $\Phi_\lambda(x) \to 1$ $(\lambda \to 0)$; therefore the probability of the process exiting all K_n is 0.

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Note: Figure translations are in progress. See original paper for figures.

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