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MATHEMATICS

L. E. EVTUSHIK

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Abstract

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MATHEMATICS

L. E. EVTUSHIK

ON ONE CONSTRUCTION OF INVARIANT FORMS AND STRUCTURAL EQUATIONS OF INFINITE GROUPS

(Presented by Academician P. S. Aleksandrov on 31 III 1962)

Let

$$\partial_k x^i = h^i_k(X^l, x^l, x^{\alpha_1}), \dots, \partial_{k_1 \dots k_p} x^i = h^i_{k_1 \dots k_p}(X^l, x^l, x^{\alpha_1}, \dots, x^{\alpha_p})$$
$$(i, k, l = 1, \dots, n) \tag{1}$$

be an involutive system of differential equations of order p, determining an infinite group of transformations

$$x^i = \varphi^i(X^k). \tag{2}$$

Here $\partial_{k_1 \dots k_s} x^i$ are the partial derivatives of the variables x^i with respect to X^k, and $x^{\alpha_1}, \dots, x^{\alpha_p}$ are parametric derivatives from the first to the p-th order, respectively.

In solving the problem of constructing invariant forms of infinite groups, É. Cartan ([1]) reduces equations (1) to a Pfaffian system and gradually transforms it into a system written by means of invariant forms. However, one can avoid the calculations and arguments that are often repeated in this procedure if one first constructs the invariant forms of the group of all analytic transformations. Such a construction is needed in itself, since this group underlies various geometric theories.

Let now

$$Y^i = a^i(X^k) = x^i + x^i_k X^k + \dots + \frac{1}{p!} x^i_{k_1 \dots k_p} X^{k_1} \dots X^{k_p} + \dots \tag{3}$$

be the totality of all transformations analytic at the point $X^k = 0$. The space of the variables $(x^i, x^i_k, \dots, x^i_{k_1 \dots k_p})$ is the representation space of the group of all analytic transformations, showing how the derivatives $\partial x^i / \partial X^k, \dots, \partial^p x^i / \partial X^{k_1} \dots \partial X^{k_p}$ transform under transformations x^i and with

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X^k preserved. We shall for the time being regard Y^i as arbitrary constants, and $x^i, x^i_k, \dots, x^i_{k_1 \dots k_p}$ as functions of a sufficiently large number of parameters. Differentiating (3) under this assumption, we find

$$dX^i = \omega^i + \omega^i_k X^k + \dots + \frac{1}{p!}\omega^i_{k_1 \dots k_p} X^{k_1} \dots X^{k_p} + \dots, \tag{4}$$

where by $\omega^i, \omega^i_k, \dots, \omega^i_{k_1 \dots k_p}$ the following differential forms are denoted:

$$\begin{gathered} \omega^i = \overset{*}{x}{}^i_k dx^k \qquad (\overset{*}{x}{}^i_k x^k_l = \delta^i_l), \\ \omega^i_k = \overset{*}{x}{}^i_l (dx^l_k - x^l_{km}\omega^m), \\ \dots\dots\dots\dots\dots\dots \\ \omega^i_{k_1 \dots k_p} = \overset{*}{x}{}^i_l \left(dx^l_{k_1 \dots k_p} - x_{lm\{k_1 \dots k_\alpha} \omega^m_{k_{\alpha+1} \dots k_p\}} \right), \end{gathered} \tag{5}$$

and the braces denote the sum over all possible permutations of pairs of indices taken one from different groups $(k_1 \dots k_\alpha)$ and $(k_{\alpha+1} \dots k_p)$, with subsequent summation over $\alpha = 1, \dots, p$. Formula (4) shows that the forms (5) remain invariant when X^k is preserved and under all analytic transformations over $Y^i = a^i(X^k)$ and the transformations induced by them-offsetof

formations $(x^i, x^i_k, \dots, x^i_{k_1 \dots k_p})$. The structural equations of the invariant forms $\omega^i, \omega^i_k, \dots, \omega^i_{k_1 \dots k_p}$ are easily obtained by exterior differentiation of (4)

$$\begin{gathered} D\omega^i = [\omega^k \omega^i_k], \qquad D\omega^i_k = [\omega^l_k \omega^i_l] + [\omega^l \omega^i_{kl}], \\ \dots\dots\dots\dots\dots\dots \\ D\omega^i_{k_1 \dots k_p} = [\omega^l_{\{k_1 \dots k_\alpha} \omega^i_{k_{\alpha+1} \dots k_p\} l}] + [\omega^l \omega^i_{k_1 \dots k_p l}]. \end{gathered} \tag{6}$$

It is necessary to note the following property of the forms (5): the invariant forms $\omega^i, \omega^i_k, \dots, \omega^i_{k_1 \dots k_p}$ take the values

$$\omega^i = dX^i, \qquad \omega^i_k = 0, \dots, \qquad \omega^i_{k_1 \dots k_p} = 0$$

if and only if

$$x^i = a^i(X^k), \qquad x^i_k = \partial_k a^i, \dots, \qquad x^i_{k_1 \dots k_p} = \partial_{k_1 \dots k_p} a^i.$$

Passing now to the group with defining equations (1), put in the forms $\omega^i, \omega^i_k, \dots, \omega^i_{k_1 \dots k_{p-1}}$

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$$x^i_k = h^i_k(X^l, x^l, x^{\alpha_1}),$$

$$\dots\dots\dots\dots\dots\dots$$

$$x^i_{k_1\dots k_p} = h^i_{k_1\dots k_p}(X^l, x^l, x^{\alpha_1}, \dots, x^{\alpha_p}).$$

Denoting by $\omega^{\alpha_1}, \dots, \omega^{\alpha_{p-1}}$ the forms, linearly independent among $\omega^i_k, \dots, \omega^i_{k_1\dots k_{p-1}}$, we obtain

$$\omega^i_k = C^i_{k\alpha_1}\omega^{\alpha_1} + \overline{C}^i_{kl}(\omega^l - dX^l),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \tag{7}$$

$$\omega^i_{k_1\dots k_{p-1}} = C^i_{k_1\dots k_{p-1}\alpha_{p-1}}\omega^{\alpha_{p-1}} + \dots + C^i_{k_1\dots k_{p-1}\alpha_1}\omega^{\alpha_1} + C^i_{k_1\dots k_{p-1}l}(\omega^l - dX^l),$$

where all coefficients are invariants of our group.

It can be shown that the defining equations (1) are equivalent to the system of Pfaffian equations

$$\omega^i = dX^i, \qquad \omega^{\alpha_1} = 0, \dots, \qquad \omega^{\alpha_{p-1}} = 0.$$

On this basis one can prove the main assertion: the group with defining equations (1) can be defined as the group of transformations of the variables $x^i, x^{\alpha_1}, \dots, x^{\alpha_{p-1}}$ $(X^k = 0)$ leaving invariant the forms

$$\omega^i, \omega^{\alpha_1}, \dots, \omega^{\alpha_{p-1}}.$$

Taking into account equations (6) and (7), we obtain the structural equations for the invariant forms $\omega^i, \omega^{\alpha_1}, \dots, \omega^{\alpha_{p-1}}$

$$D\omega^i = \frac{1}{2}C^i_{kl}[\omega^k\omega^l] + C^i_{k\alpha_1}[\omega^k\omega^{\alpha_1}],$$

$$D\omega^{\alpha_1} = \frac{1}{2}C^{\alpha_1}_{A_1B_1}[\omega^{A_1}\omega^{B_1}] + C^{\alpha_1}_{k\alpha_2}[\omega^k\omega^{\alpha_2}], \tag{8}$$

$$\dots\dots\dots\dots\dots\dots\dots$$

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$$D\omega^{\alpha_{p-1}} = \frac{1}{2} C^{\alpha_{p-1}}_{A_{p-1}B_{p-1}} [\omega^{A_{p-1}} \omega^{B_{p-1}}] + C^{\alpha_{p-1}}_{k\alpha_p} [\omega^k \omega^{\alpha_p}],$$

where the indices A_s, B_s run through the group of indices $i, \alpha_1, \dots, \alpha_s$. It also follows from equations (6) and (7) that some of the coefficients of equations (8) are known in advance to be zero. Thus,

$$C^{\alpha_r}_{\beta_s \gamma_t} = 0 \quad \text{for } s + t > r + 1.$$

Moscow State University
named after M. V. Lomonosov

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CITED LITERATURE

1. E. Cartan, *Sur la structure des groupes infinis de transformations*, Oeuvres complètes, partie II, 2, Paris, 1953, p. 571; *Les groupes de transformations continus, infinis, simples*, ibid., p. 857.

Note: Figure translations are in progress. See original paper for figures.

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