Reports of the Academy of Sciences of the USSR

PHYSICS

1962

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR
1962. Volume 143, No. 5

PHYSICS

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ON THE QUESTION OF THE STABILITY OF A PLASMA CYLINDER IN THE CASE OF A NONUNIFORM DISTRIBUTION OF CURRENT OVER THE CROSS SECTION

(Presented by Academician B. P. Konstantinov, 25 XII 1961)

We shall study the stability of a plasma cylinder of radius a, assuming that the equilibrium distribution of the magnetic field inside the cylinder and outside it in vacuum has the form:

$$\frac{\mathbf{H}}{H_0} = \begin{cases} \mathbf{i}_\varphi g\frac{r}{d} + \mathbf{i}_z h, & \text{for } 0 \le r \le d, \\ \mathbf{i}_\varphi \frac{r}{a}\left(g + f\ln\frac{r}{d}\right) + \mathbf{i}_z h\left(1 + l\ln\frac{r}{d}\right), & \text{for } d \le r \le a, \\ \mathbf{i}_\varphi \frac{a}{r} + \mathbf{i}_z h\left(1 + l\ln\frac{a}{d}\right), & \text{for } a \le r < \infty. \end{cases} \quad (1)$$

Here $g = 1 - f\ln\frac{a}{d}$, $\mathbf{i}_r, \mathbf{i}_\varphi, \mathbf{i}_z$ are the unit vectors of the cylindrical coordinate system; d, f, g, h, l, H_0 are constants such that the pressure $p \ge 0$.

We shall assume that the medium inside the cord may be regarded as incompressible, inviscid, and perfectly conducting. We take the dependence of a perturbation on time and on φ, z in the form $e^{i(\omega t + m\varphi + kz)}$. Introducing the displacement $\vec{\xi} = \mathbf{v}/i\omega$ and the perturbation of the total pressure (hydrodynamic plus magnetic) P^*, we write the initial linearized system of equations of magnetohydrodynamics in the form

$$\frac{H_0^2}{a^2}(s^2 - \Omega^2)\vec{\xi} = -4\pi\nabla P^* + \mathbf{i}_r\xi_r\frac{d}{dr}\left(\frac{H_\varphi}{r}\right)^2 - \frac{2isH_0H_\varphi}{ar}(\mathbf{i}_r\xi_\varphi - \mathbf{i}_\varphi\xi_r); \quad (2)$$

$$\operatorname{div}\vec{\xi} = 0, \quad (3)$$

where

$$s = \frac{a}{H_0}\left(\frac{mH_\varphi}{r} + kH_z\right), \qquad \Omega^2 = \frac{4\pi\rho a^2\omega^2}{H_0^2}.$$

We restrict ourselves to the study of long-wavelength perturbations with $m \geq 1$, so that $|ka| \ll 1$. We shall neglect k^2r^2 in comparison with m^2; then, as is not difficult to show, the term with ξ_z in equation (3) may be omitted.* Expressing all unknowns through $X = r\xi_r$ and substituting in (2)–(3) the distribution (1) for the region $d \leq r \leq a$, we obtain

$$(s^2 - \Omega^2)\frac{d^2X}{ds^2} + 2s\frac{dX}{ds} - \left\{\frac{s^2 - \Omega^2}{(b+f)^2} + \frac{M + Ns}{b+f}\right\} X = 0, \tag{4}$$

where

$$M = \frac{2b^2(lg - f)l}{(b+f)^2}, \qquad N = \frac{2f(2b+f)}{m(b+f)^2}, \qquad b = \frac{kahl}{m},$$

$$s = s_0 + m(b+f)\ln\frac{r}{d}, \qquad s_0 = mg + kah,$$

and it is assumed that $b + f \neq 0$.†

* In the present work the case is not studied in which, in the interval under consideration, there is a value $s^2 = \Omega^2$. Near such a point various small terms discarded in the initial equations of magnetohydrodynamics may be significant.

The boundary conditions for $X(s)$ are found from the matching condition with the solution in the region $r \leqslant d$, where $X \sim r^m$, and from the condition of equality of the total pressures on the perturbed surface of the column:

$$\left(\frac{d\ln X}{ds}\right)_{\substack{r=d\\ s=s_0}} = \frac{1}{b+f}; \tag{5}$$

$$\left(\frac{d\ln X}{ds}\right)_{\substack{r=a\\ s=s_1}} = \frac{s_1(2 - s_1)}{(b+f)(s_1^2 - \Omega^2)}. \tag{6}$$

Let us first consider the stability of a cylinder with a homogeneous longitudinal field ($l = 0$) with respect to perturbations $m = 1$. For $\Omega^2 < 0$, solution (4) has the form

$$X = e^{s/f}\left\{A_1 + A_2\int_{s_0/f}^{s/f}\frac{e^{-2y}\,dy}{y^2 - \Omega^2/f^2}\right\},$$

$$A^i = \text{const}.$$

Fig. 1

Figure 1: Fig. 1

With the aid of (5)–(6) we obtain

$$\Omega^2 = 2s_1(s_1 - 1) =$$

$$= 2kah(1 + kah). \tag{7}$$

The expression for the oscillation frequency does not depend on the parameter f, which characterizes the current distribution over the cross section. Formula (7) coincides with that which holds for $f = 0$ [1].

Fig. 1

We now turn to the analysis of stability for the case of a longitudinal current homogeneous over the cross section ($f = 0,\ l \neq 0$). For the boundary of the stability region ($\Omega = 0$), taking into account that the solutions of (4) are* $\frac{1}{\sqrt{s}} I_{\pm\nu}\left(\frac{s}{b}\right)$, where

$$\nu = \frac{1}{2}\sqrt{1 + \frac{8}{b}},$$

we find

$$\left\{ I'_\nu\left(\pm\frac{s_1}{b}\right) \pm \left[1 - \frac{(4\nu^2+1)b}{4s_1}\right] I_\nu\left(\pm\frac{s_1}{b}\right) \right\} \left\{ K'_\nu\left(\pm\frac{s_0}{b}\right) + \left(1 + \frac{b}{2s_0}\right) K_\nu\left(\pm\frac{s_0}{b}\right) \right\} =$$

$$= \left\{ K'_\nu\left(\pm\frac{s_1}{b}\right) \pm \left[1 - \frac{(4\nu^2+1)b}{4s_1}\right] K_\nu\left(\pm\frac{s_1}{b}\right) \right\} \times$$

$$\times \left\{ I'_\nu\left(\pm\frac{s_0}{b}\right) \mp \left(1 + \frac{b}{2s_0}\right) I_\nu\left(\pm\frac{s_0}{b}\right) \right\}. \tag{8}$$

Here the prime denotes differentiation with respect to the argument of the Bessel functions, and it is assumed that the point $s = 0$ does not lie inside the interval s_0, s_1, so that the sign should be chosen from the condition $\pm s_i/b > 0$.

Equation (8) determines a one-parameter family of curves $s_0(b)$, where $s_0 = m + kah$, $b = kahl/m$. As the parameter one may take the quantity $m\ln(a/d) = (s_1 - s_0)/b$. For $a = d$, instability occurs in the band $0 < s_0 < 1$, which agrees with the result following from the formulas of paper [1]. For the case $m\ln(a/d) = 1/4$, the curves $s_0(b)$ are shown in Fig. 1

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[*] For arbitrary f and l, and $\Omega = 0$, $X(s)$ is expressed in terms of a degenerate hypergeometric function.

(we are speaking of those regions where the signs of s_0 and s_1 are the same). The numbers below the abscissa axis give the values of the parameter ν. The region of instability in the figure is shaded.

Serious difficulties arise in the analysis of stability in the case when $s(r)$ in the interval $d \ll r \ll a$ passes through zero. In order that the initial equation (4) have no singularities, it is necessary to take $\Omega^2 < 0$.

For $f = 0$ and $s^2 \ll b^2$, the solutions of (4) are spherical Legendre functions defined in the complex plane s/Ω with cuts from $-\infty$ to -1 and from 1 to ∞. Putting

$$\mu = \frac{1}{2}\sqrt{1 + \frac{8}{b} - \frac{4\Omega^2}{b^2}},$$

we obtain

$$X = B_1 P_{-1/2+\mu}\left(\frac{s}{\Omega}\right) + B_2 P_{-1/2+\mu}\left(-\frac{s}{\Omega}\right), \qquad B_i = \text{const.} \tag{9}$$

Let us first assume that μ is not close to an integer and that the oscillation frequency satisfies the condition $(-\Omega^2) \ll s_{\max}^2 \ll b^2$. Let us find how $X(s)$ changes on passing through the point $s = 0$. If for $s < 0$, $|s/\Omega| \gg 1$, the solution has the form

$$X = C_1(-s)^{\mu-1/2} + C_2(-s)^{-\mu-1/2}, \qquad C_i = \text{const},$$

then for values $s > 0$, $|s/\Omega| \gg 1$, with the aid of (9) we find

$$X = \frac{s^{\mu-1/2}}{\sin\pi\mu}\left[C_1 - C_2\left(-\frac{\Omega^2}{4}\right)^{-\mu}\frac{\Gamma(\mu)\Gamma(-\mu+1/2)}{\Gamma(-\mu)\Gamma(\mu+1/2)}\cos\pi\mu\right] - \frac{s^{-\mu-1/2}}{\sin\pi\mu}\left[C_2 - C_1\left(-\frac{\Omega^2}{4}\right)^{\mu}\frac{\Gamma(-\mu)\Gamma(\mu+1/2)}{\Gamma(\mu)\Gamma(-\mu+1/2)}\cos\pi\mu\right].$$

One of the constants C_1 or C_2 may be taken equal to unity. For sufficiently small $|\Omega^2|$ there is no dependence on the ratio C_2/C_1, and stability is determined only by the behavior of $X(s)$ near $s = 0$. For imaginary μ, because of the strong oscillation, one can always find solutions corresponding to small negative Ω^2—a result agreeing with Suydam' s theorem (^{2}). This region is shaded in the figure. For real μ, in accordance with theorem (^{2}), the solutions under consideration do not exist in the limiting case $|\Omega^2| \to 0$.*

With increasing oscillation frequency $|\Omega|$, the situation may change. Suppose, for example, that $|s_0| \ll 1$, $|s_1| \ll 1$, and that Ω^2 is of the order of s_0 and s_1. Substituting (9) into conditions (5)–(6), we find

$$\Omega^2 = -2s_0. \tag{10}$$

It follows from this that in the region $0 < \nu < 1/2$, stable according to Suydam' s criterion (^{2}), unstable oscillations may in fact arise.

If the oscillation frequency is so large that the conditions

$$\left|\frac{b}{s^2 - \Omega^2}\right| \ll 1, \qquad \left|\frac{bs}{s^2 - \Omega^2}\right| \ll 1, \qquad |b| \ll 1,$$

are satisfied, then the solutions of (4) for $f = 0$ are written as series in powers

$$e^{\pm s/b} \frac{1}{\sqrt{s^2 - \Omega^2}} \left(\frac{s - \Omega}{s + \Omega}\right)^{\pm 1/2\,\Omega} \left\{1 \mp \frac{b}{2} \int \frac{1 + \Omega^2 \mp 2s}{(s^2 - \Omega^2)}\, ds + \ldots\right\},$$

and with the aid of (5)–(6) we find

$$\begin{aligned} \Omega^2 &= 2s_1(s_1 - 1) - b(s_1 - 1) \\ &\quad + b(s_0 - 1)\frac{s_1^2 - \Omega^2}{s_0^2 - \Omega^2}\left(\frac{s_1 + \Omega}{s_1 - \Omega}\frac{s_0 - \Omega}{s_0 + \Omega}\right)^{1/\Omega}\left(\frac{a}{d}\right)^{-2m} + \ldots \end{aligned} \tag{11}$$

* It should be noted that for small oscillation frequencies near the point $s = 0$, the nonideal conductivity of the plasma may be important.

Let $s_0 s_1 < 0$ and $mb \ln(a/d)$ be of order unity. For small b, formula (11) gives a distribution of the regions of stability and instability opposite to that given by Suydam' s theorem ^{2}. For example, if $s_0 = -s_1$, $b > 0$, $mb \ln(a/d) < 2$, $m \ln(a/d) \gg 1$, then $\Omega^2 < 0$, whereas according to Suydam' s criterion ^{2} one should have $\Omega^2 > 0$.

Thus, the picture shown in Fig. 1 is incomplete, since the curve corresponding to this negative value of Ω^2 does not necessarily lie inside the instability region hatched in the figure.

Let us also consider the transition to a filament with a surface current, when $b + f \gg 1$, $a \simeq d$, and $(b + f)\ln(a/d)$ is a finite quantity. We assume that $\ln(-\Omega^2) \ll (b + f)$. Taking the principal terms into account,

$$X = D_1 \left\{1 + \frac{2}{2(b + f)}\left[M \ln\left(1 - \frac{s^2}{\Omega^2}\right) + Ns\right]\right\} + D_2 \operatorname{arc\,tg} \frac{is}{\Omega}, \qquad D_i = \text{const}.$$

Substituting into conditions (5)–(6), we find

$$\Omega^2 = s_1^2 + s_0^2 - 2gs_0 - m(1 - g^2). \tag{12}$$

In the present case, irrespective of whether $s(r)$ passes through zero or not, we obtain the formula of Shafranov' s work ^{1}.*

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Received
6 XII 1961

CITED LITERATURE

[1] V. D. Shafranov, *Plasma Physics and the Problem of Controlled Thermonuclear Reactions*, 4, Publishing House of the Academy of Sciences of the USSR, 1958, p. 61.
[2] B. R. Suydam, Proc. II International Conference on the Peaceful Uses of Atomic Energy, Geneva, 1958, 1, *Physics of Hot Plasma and Thermonuclear Reactions*, 1959, p. 89.

[*] Integrating (2)–(3), one can show that formula (12) remains valid for an arbitrary distribution of the field $H(r)$, if $d/dr \gg 1/r$, $\Omega^2 > 0$, and $|\Omega|$ is not close to zero.

Note: Figure translations are in progress. See original paper for figures.

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