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V. S. BUSLAEV

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Abstract

Full Text

MATHEMATICAL PHYSICS

V. S. BUSLAEV

ON THE SHORT-WAVE ASYMPTOTICS IN THE PROBLEM OF DIFFRACTION BY CONVEX BODIES

(Presented by Academician V. I. Smirnov, 12.III.1962)

1. In the present note a rigorous justification is obtained for the principal terms of the short-wave asymptotics of the Green function $G(x', x; k)$ of the exterior Dirichlet problem for the Helmholtz equation in the plane. The Green function satisfies the equation

$$\begin{aligned} (-\nabla_x^2 - k^2)G(x', x; k) &= \delta(x - x') \\ (x, x' \in D, k^2 > 0) \end{aligned} \quad (1)$$

and the boundary conditions

$$G(x', x; k)|_{x \in L} = 0;$$

$$\int_{\Sigma_R} dS_x \left| \frac{\partial G(x', x; k)}{\partial |x|} - ikG(x', x; k) \right|^2 \rightarrow 0, \quad |x| = R \rightarrow \infty. \quad (2)$$

The domain D is situated outside a finite closed convex contour L . It is assumed about the contour that its radius of curvature $\rho(s)$, as a function of the arc length s , has two continuous derivatives and that $\rho(s) \geq \rho_0 > 0$.

2. A large number of works, mainly physical ones,* have been devoted to the construction of the short-wave asymptotics (asymptotics as $k \rightarrow +\infty$) of the function $G(x', x; k)$. However, even for the principal terms of the formulas derived in these works, an exact justification is lacking. Such a justification, obtained in the present note, required a further improvement of the asymptotic formulas. Namely, in Sec. 3 an asymptotic $Q(x', x; k)$ of the Green function is given for an arbitrary location of the points x' and x in the domain D ; this asymptotic has the following properties: a) $Q(x', x; k)$, as a function of x , is

continuous in D except at the point x' , where it has the singularity required for the Green function; b) the function $Q(x', x; k)$ is symmetric,

$$Q(x', x; k) = Q(x, x'; k),$$

and exactly satisfies the conditions (2); c) the corresponding residual

$$K(y, x; k) \equiv (-\nabla_y^2 - k^2)Q(y, x; k) - \delta(y - x) \quad (3)$$

is a continuous function of y , except for the geometric boundary of light and shadow, where $K(y, x; k)$ has discontinuities of the first kind and singularities of the type of δ -functions, proportional to the jumps of $\nabla_y Q(y, x; k)$.

The estimate of the error of the asymptotics, and thereby its justification, is carried out according to the following scheme. The relation

$$G(x', x; k) + \int_D dy G(x', y; k)K(y, x; k) = Q(x', x; k) \quad (4)$$

is valid.

* See, for example, the survey (1). We also note here the works (2-5), in which the asymptotics is constructed in the form of contour integrals of a special kind and which are directly related to the subsequent exposition.

It can be considered as an equation for the function $G(x', x; k)$ (x' fixed). The operator K in this equation will be small in a certain space as $k \rightarrow +\infty$, which gives the corresponding estimate for $G(x', x; k) - Q(x', x; k)$ (item 4).

Let us note here that, in the study of the operator K , many properties of contour integrals of type (10) (see below), studied by V. A. Fock^(2,5-7), turned out to be essential.

3. To write the asymptotics of $Q(x', x; k)$, "phases" are introduced—the lengths of certain lines joining the points x' and x . The segments T_+ and T_- (see Fig. 1), tangent to the contour L and drawn through the source point x' , divide the domain D into two parts: $R(x')$ (illuminated region) and $S(x')$ (shadow). For $x \in R(x')$, three phases are considered: $\Phi_0 = |x - x'|$ (the phase of the incident wave); Φ_R , the length of the broken line $x'Px$, "reflected" from the contour according to the laws of geometrical optics (the phase of the reflected wave); and Φ_s , the length of the shortest curve $x'H_-HH_+x$ enveloping L (the phase of the creeping wave). In the region $S(x')$, two phases (of creeping waves) are considered—the lengths of the shortest curves enveloping the contour. To each phase in the asymptotics of $Q(x', x; k)$ there corresponds a certain term.

Fig. 1

Figure 1: Fig. 1

Fig. 1

To the phase Φ_0 of the incident wave there corresponds $\frac{i}{4}H_0^{(1)}(k\Phi_0)$.

To the phase Φ_R of the reflected wave there corresponds

$$\frac{i}{4}H_0^{(1)}(k\Phi_R)\Phi_R^{1/2}\frac{M^{1/2}(s)}{R^{1/2}(s)}V^{(R)}(X,Y,Y'). \quad (5)$$

Here $M(s) = (k\rho(s)/2)^{1/3}$, s is the length of the arc corresponding to the point of reflection P ;

$$\begin{aligned} V^{(R)}(X,Y,Y') &= V^{(R)}(X,Y',Y) = \\ &= -\frac{e^{-i\pi/4}}{\pi^{1/2}} \exp \left[-i\zeta_R X - i\frac{2}{3}(Y - \zeta_R)^{3/2} - i\frac{2}{3}(Y' - \zeta_R)^{3/2} + i\frac{4}{3}(-\zeta_R)^{3/2} \right] \times \\ &\quad \times \int_{-\infty}^{\infty} d\xi e^{i\xi X} w_1(\xi - Y) \frac{v(\xi)}{w_1(\xi)} w_1(\xi - Y'); \end{aligned} \quad (6)$$

$$(-\zeta_R)^{1/2} = M(s) \cos \theta;$$

θ is the angle of reflection; $w_1(\zeta)$ and $w_2(\zeta)$ are Airy functions in the definition of V. A. Fock; $v(\zeta) = \frac{1}{2i}(w_1(\zeta) - w_2(\zeta))$;

$$X = \frac{M(s)}{\rho(s)}\Phi_R,$$

$$Y = \frac{M^2(s)}{\rho^2(s)}t(t + 2\rho(s)\cos\theta), \quad (7)$$

$$Y' = \frac{M^2(s)}{\rho^2(s)}t'(t' + 2\rho(s)\cos\theta);$$

t is the length of the segment Px , and t' is the length of the segment $x'P$.

The function $V^{(R)}(X,Y,Y')$ satisfies the equation

$$4 \frac{M^4(s)}{\rho^2(s)} V_{YY}^{(R)} + ik (2\nabla\Phi_R \nabla V^{(R)} + V^{(R)} \nabla^2 \Phi_R) = 0, \quad (8)$$

where $V_{YY}^{(R)}(X, Y, Y')$ denotes the second derivative with respect to Y for fixed X and Y' , and ∇ corresponds to differentiation with respect to x for fixed x' .

Finally, to the phase Φ_s of the creeping wave there corresponds

$$\frac{i}{4} H_0^{(1)}(k\Phi_s) \Phi_s^{1/2} \left(\frac{M(s)M(s')}{\rho(s)\rho(s')} \right)^{1/4} V(X, Y, Y'), \quad (9)$$

where

$$V(X, Y, Y') = V(X, Y', Y) = \frac{e^{i\pi/4}}{2\pi^{1/2}} \exp \left[-i\frac{2}{3}Y^{3/2} - i\frac{2}{3}Y'^{3/2} \right] \times \\ \times \int d\xi e^{i\xi X} w_1(\xi - Y') \left[w_2(\xi - Y) - \frac{w_2(\xi)}{w_1(\xi)} w_1(\xi - Y) \right] \quad (Y' \geq Y), \quad (10)$$

the contour of integration enclosing the poles of the integrand situated on the ray $\arg \xi = \pi/3$;

$$Y = \frac{M^2(s)}{\rho^2(s)} t^2, \\ Y' = \frac{M^2(s')}{\rho^2(s')} t'^2, \quad (11)$$

$$X = Y'^{1/2} + Y^{1/2} + \int_{s'}^s ds \frac{M(s)}{\rho(s)};$$

here the integral is taken along the arc H_-HH_+ ; s corresponds to the point H_+ , and s' to the point H_- ; t is the length of the segment H_+x , and t' the length of $x'H_-$.

The function $V(X, Y, Y')$ satisfies the equation

$$4 \frac{M^4(s)}{\rho^2(s)} V_{YY} + ik (2\nabla\Phi_s V + V \nabla^2 \Phi_s) = 0, \quad (12)$$

in which the differentiation is performed analogously to (8).

4. For the asymptotics of $Q(x', x; k)$ —the right-hand side of equation (4)—one easily obtains:

$$|Q(x', x; k)| \leq C \frac{1 + |\ln k| |x - x'|}{[1 + k|x - x'| \ln^2 k |x - x'|]^{1/2}}. \quad (13)$$

The operator K in this equation, considered on the space of continuous functions (except, possibly, at the point x') admitting an estimate of type (13), has small norm as $k \rightarrow +\infty$:

$$\|K\| \leq \alpha \max_{s \in L} M^{-1+\varepsilon}(s) \quad (\varepsilon > 0) \quad (14)$$

(the norm is determined by the smallest constant C in an inequality of type (13)).

For the error of the asymptotic formulas it follows immediately from this that

$$\|G(x', x; k) - Q(x', x; k)\| \leq \alpha_1 \max_{s \in L} M^{-1+\varepsilon}(s). \quad (15)$$

This inequality gives the justification of the short-wave asymptotics.

The constants α and α_1 in (14) and (15) depend only on the properties of the contour. The author expresses deep gratitude to Prof. O. A. Ladyzhenskaya and L. D. Faddeev for their attention to the work and valuable advice.

Leningrad State University
named after M. V. Lomonosov

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