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## Abstract

## Full Text

## GEOPHYSICS

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# REDUCTION OF A SYSTEM OF DIFFERENTIAL EQUATIONS USED IN SHORT-RANGE WEATHER FORECASTING TO A SYSTEM OF ALGEBRAIC EQUATIONS

In short-range weather forecasting, solutions of a simplified system of differential equations of hydrodynamics are used; this system can be written in the form (1):

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uv}{\partial y}+\frac{\partial u\omega}{\partial \zeta}=-\frac{\partial \Phi}{\partial x}+lv; \tag{1}$$

$$\frac{\partial v}{\partial t}+\frac{\partial uv}{\partial x}+\frac{\partial v^2}{\partial y}+\frac{\partial v\omega}{\partial \zeta}=-\frac{\partial \Phi}{\partial y}-lu; \tag{2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \omega}{\partial \zeta}=0; \tag{3}$$

$$\frac{\partial \Gamma}{\partial t}+\frac{\partial u\Gamma}{\partial x}+\frac{\partial v\Gamma}{\partial y}+\frac{\partial \omega\Gamma}{\partial \zeta}+c^2\omega=0. \tag{4}$$

Here $x, y$ are horizontal coordinates; $\zeta$ is the reduced pressure; $t$ is time; $u, v$ are the velocity components along the $X, Y$ axes; $\omega$ is a quantity proportional to the vertical velocity; $\Phi$ is the deviation of the geopotential from the standard value; $l$ is the Coriolis parameter; $\Gamma = \zeta^2\partial\Phi/\partial\zeta$ is a quantity proportional to the deviation of temperature from the standard value; $c$ has the dimension of velocity and is connected with the atmospheric "stability" parameter. As boundary conditions one may take the vanishing of $\omega$ at $\zeta = 1$ (the Earth) and at $\zeta = 0$ (the upper boundary of the atmosphere).

In the coordinate plane $(X, Y)$ let us select a rectangular domain $0 \le x \le L_1,\ 0 \le y \le L_2$; in this domain consider a grid of points with coordinates

$$x_i = \frac{1}{p} i L_1 \quad (i = 0, 1, 2, \dots, p), \qquad y_j = \frac{1}{q} j L_2 \quad (j = 0, 1, 2, \dots, q).$$

Let us also divide the atmosphere by height into $h$ layers with boundaries at

$$\zeta = \zeta_k = \frac{1}{h} k \quad (k = 0, 1, 2, \dots, h).$$

Introduce the notation

$$F(x_i, y_j, \zeta_k, t) = F_{ijk}(t).$$

We approximate the values of all the unknown functions $F$ by finite sums of the form*

$$F(x, y, \zeta, t) = \sum_{m=0}^{p} \sum_{n=0}^{q} \sum_{s=0}^{h} \overline{F}_{mns}(t) \left(\frac{x}{L_1}\right)^m \left(\frac{y}{L_2}\right)^n \zeta^s, \tag{5}$$

where the $\overline{F}_{mns}$ are determined so that, at the nodes of the grid chosen by us, sums of the type (5) give exact values. Let now

$$f(x) = f(0) + \sum_{m=1}^{p} \overline{f}_m \left(\frac{x}{L_1}\right)^m. \tag{6}$$

Let us call

$$\int_0^x f(x)\, dx = \overline{f(x)} \quad \text{and} \quad \int_0^{x_i} f(x)\, dx = \overline{f}_i.$$

Then

$$\overline{f}_i = \frac{1}{p} i L_1 f_0 + \sum_{m=1}^{p} \frac{L_1}{m+1} \overline{f}_m \left(\frac{i}{p}\right)^{m+1}.$$

Writing these relations $p$ times $(i = 1, 2, \dots, p)$,

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* Instead of powers of $x, y, \zeta$, trigonometric or any other functions may be taken.

we obtain $p$ equations for determining the $p$ quantities $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_p$ in terms of $\bar{f}_i$ and $f_0$. Substituting these expressions into (6) and collecting terms, we obtain

$$f_i = P_0^i f_0 + \sum_{m=1}^{p} P_m^i \bar{f}_m. \tag{7}$$

Here $P_0^i, P_m^i$ can be computed once and for all (they depend on $p$)*.

Let us transform the system (1)–(4) of the equations of hydrodynamics, analogously to how this is done in ([2]). Namely, let us integrate both sides of (1) partially with respect to $x$ from 0 to $x$. We obtain

$$\overline{\frac{\partial u}{\partial t}+\frac{\partial uv}{\partial y}+\frac{\partial u\omega}{\partial \xi}}-lv=(\Phi)_{x=0}-\Phi+(u^2)_{x=0}-u^2. \tag{8}$$

Now suppose–and this is the main point of our reasoning–that not only $u, v$, but also their products $u^2, uv, \ldots$ are representable by finite sums (5). Then, instead of (8), we can write (we omit the indices $j$ and $k$):

$$\left(\overline{\frac{\partial u}{\partial t}+\frac{\partial uv}{\partial y}+\frac{\partial u\omega}{\partial \xi}-lv}\right)_i=\Phi_0-\Phi_i+u_0^2-u_i^2,$$

or, by (7):

$$\left(\frac{\partial u}{\partial t}+\frac{\partial uv}{\partial y}+\frac{\partial u\omega}{\partial \xi}-lv\right)_i=$$

$$=P_0^i\left(\frac{\partial u}{\partial t}+\frac{\partial uv}{\partial y}+\frac{\partial u\omega}{\partial \xi}-lv\right)_0+\frac{1}{L_1}\sum_{m=1}^{p}P_m^i[\Phi_0-\Phi_m+(u^2)_0-(u^2)_m]. \tag{9}$$

Then carrying out the integration with respect to $y$ and $\xi$, we finally obtain, instead of equation (1), the system of equations:

$$\frac{du_{ijk}}{dt}=-\frac{1}{L_1}\sum_{m=1}^{p}P_m^i(\Phi_{mjk}+u_{mjk}^2)-\frac{1}{L_2}\sum_{n=1}^{q}Q_n^j u_{ink}v_{ink}-\sum_{s=1}^{h}S_s^k u_{ijs}\omega_{ijs}+(lv)_{ijk} \tag{10}$$

for $i=1,2,\ldots,p-1$; $j=1,2,\ldots,q-1$; $k=1,2,\ldots,h$ ($Q$ and $S$ are obtained from $P$ by replacing $p$ by $q$ and $h$, respectively). Here, for simplicity of exposition, we have set $u_{ijk}, v_{ijk}, \omega_{ijk}, \Phi_{ijk}$ equal to zero if $i=0$, or $i=p$, or $j=0$, or $j=q$ (the boundaries of the region), and also when $k=0$ (we are interested in the initial-data problem).

In the same way, from (2) we obtain

$$\frac{dv_{ijk}}{dt}=-\frac{1}{L_1}\sum_{m=1}^{p}P_m^i u_{mjk}v_{mjk}-\frac{1}{L_2}\sum_{n=1}^{q}Q_n^j(\Phi_{ink}+v_{ink}^2)-\sum_{s=1}^{h}S_s^k v_{ijs}\omega_{ijs}-(lu)_{ijk}, \tag{11}$$

$$i=1,2,\ldots,p-1; \quad j=1,2,\ldots,q-1; \quad k=1,2,\ldots,h.$$

* Thus, for $p = 1$ we have $P_0^1 = -1$, $P_1^1 = 2$; for $p = 2$, $P_0^1 = -0.5$, $P_1^1 = 2$, $P_2^1 = 0.5$, $P_0^2 = 1$, $P_1^2 = -8$, $P_2^2 = 4$; for $p = 3$, $P_0^1 = -\frac{1}{3}$, $P_1^1 = \frac{8}{2}$, $P_2^1 = \frac{3}{2}$, $P_3^1 = -\frac{1}{6}$, $P_0^2 = \frac{1}{3}$, $P_1^2 = -6$, $P_2^2 = 3$, $P_3^2 = \frac{2}{3}$, $P_0^3 = -1$, $P_1^3 = \frac{27}{2}$, $P_2^3 = -\frac{27}{2}$, $P_3^3 = \frac{13}{2}$; for $p = 4$, $P_0^1 = -\frac{1}{4}$, $P_1^1 = \frac{2}{3}$, $P_2^1 = 3$, $P_3^1 = -\frac{2}{3}$, $P_4^1 = \frac{1}{12}$, $P_0^2 = \frac{1}{6}$, $P_1^2 = -\frac{16}{3}$, $P_2^2 = 2$, $P_3^2 = \frac{16}{9}$, $P_4^2 = -\frac{1}{6}$, $P_0^3 = -\frac{1}{4}$, $P_1^3 = 6$, $P_2^3 = -9$, $P_3^3 = \frac{14}{3}$, $P_4^3 = \frac{3}{4}$, $P_0^4 = 1$, $P_1^4 = -\frac{64}{3}$, $P_2^4 = 24$, $P_3^4 = -\frac{64}{3}$, $P_4^4 = \frac{28}{3}$.

The continuity equation will be brought to the form

$$\frac{1}{L_1}\sum_{m=1}^{p} P_m^i u_{mjk} + \frac{1}{L_2}\sum_{n=1}^{q} Q_n^j v_{ink} + \sum_{s=1}^{h} S_s^k \omega_{ijs} = 0, \tag{12}$$

$$i = 1, 2, \ldots, p-1; \quad j = 1, 2, \ldots, q-1; \quad k = 1, 2, \ldots, h-1.$$

Equation (4) will be replaced by the relation:

$$\frac{d\Gamma_{ijk}}{dt} + (c^2\omega)_{ijk} = -\frac{1}{L_1}\sum_{m=1}^{p} P_m^i u_{mjk}\Gamma_{mjk}$$

$$-\frac{1}{L_2}\sum_{n=1}^{q} Q_n^j v_{ink}\Gamma_{ink} - \sum_{s=1}^{h} S_s^k \omega_{ijs}\Gamma_{ijs}, \tag{13}$$

$$i = 1, 2, \ldots, p-1; \quad j = 1, 2, \ldots, q-1; \quad k = 1, 2, \ldots, h.$$

We have thus obtained a system of ordinary differential equations for determining, as functions of time, the values of our unknown functions $u, v, \omega, \Phi$ at the nodes of the chosen grid. It is easy to see that, with the notation we have chosen, the number of equations (12)–(13) coincides with the number of the functions themselves. Moreover, by virtue of the definition of $\hat{\Gamma}$, the values of the functions $\Phi$ and $\Gamma$ are connected by the simple relations:

$$\Gamma_{ijk} = \zeta_k \left(-\Phi_{ijk} + \sum_{s=1}^{h} S_s^k \zeta_s \Phi_{ijs}\right), \tag{14}$$

$$i = 1, 2, \ldots, p-1; \quad j = 1, 2, \ldots, q-1; \quad k = 1, 2, \ldots, h.$$

Let us now show how to reduce our system to a system of algebraic equations.

Suppose that we wish to predict the values of the meteorological quantities for a time interval $T$ (for example, one day) ahead. We divide $T$ into $\vartheta$ intervals.

Consider the time instants $t_\tau = \tau T/\vartheta$ $(\tau = 0, 1, 2, \dots, \vartheta)$ and assume that any of the functions of interest to us can be approximated on the interval $T$ in the form of a polynomial of degree $\vartheta$:

$$f(t) = f(0) + \sum_{\lambda=1}^{\vartheta} \bar{f}_\lambda \left(\frac{t}{T}\right)^\lambda .$$

Then we can again write, according to (7) $(f(t_\tau) = f^\tau)$:

$$f^\tau = \theta_0^\tau f^0 + \sum_{\lambda=1}^{\vartheta} \theta_\lambda^\tau \bar{f}_\lambda \tag{15}$$

$(\theta_0^\tau, \theta_\lambda^\tau$ are obtained from $P$ by replacing $p$ by $\vartheta)$.

Integrating (10)–(13) with respect to time, we finally obtain, taking (15) into account:

$$\begin{gathered}\frac{1}{L_1}\sum_{m=1}^{p} P_m^i \left[\Phi_{mjk}^\tau - \theta_0^\tau \Phi_{mjk}^0 + (u^2)_{mjk}^\tau - \theta_0^\tau (u^2)_{mjk}^0\right] + \\ + \frac{1}{L_2}\sum_{n=1}^{q} Q_n^j \left(u_{ink}^\tau v_{ink}^\tau - \theta_0^\tau u_{ink}^0 v_{ink}^0\right) + \sum_{s=1}^{h} S_s^k \left(u_{ijs}^\tau \omega_{ijs}^\tau - \theta_0^\tau u_{ijs}^0 \omega_{ijs}^0\right) + \\ + \frac{1}{T}\sum_{\lambda=1}^{\vartheta} \theta_\lambda^\tau \left(u_{ijk}^\lambda - u_{ijk}^0\right) - l_{ij}\left(v_{ijk}^\tau - \theta_0^\tau v_{ijk}^0\right) = 0 \quad (\tau = 1, 2, \dots, \vartheta);\end{gathered} \tag{16}$$

$$\begin{gathered}\frac{1}{L_1}\sum_{m=1}^{p} P_m^i \left(u_{mjk}^\tau v_{mjk}^\tau - \theta_0^\tau u_{mjk}^0 v_{mjk}^0\right) + \\ + \frac{1}{L_2}\sum_{n=1}^{q} Q_n^j \left[\Phi_{ink}^\tau - \theta_0^\tau \Phi_{ink}^0 + \left(v^2\right)_{ink}^\tau - \theta_0^\tau \left(v^2\right)_{ink}^0\right] + \\ + \sum_{s=1}^{h} S_s^k \left(v_{ijs}^\tau \omega_{ijs}^\tau - \theta_0^\tau v_{ijs}^0 \omega_{ijs}^0\right) + \frac{1}{T}\sum_{\lambda=1}^{\vartheta} \theta_\lambda^\tau \left(v_{ijk}^\lambda - v_{ijk}^0\right) + l_{ij}\left(u_{ijk}^\tau - \theta_0^\tau u_{ijk}^0\right) = 0\end{gathered} \tag{17}$$

$$(\tau = 1, 2, \dots, \vartheta);$$

$$\frac{1}{L_1}\sum_{m=1}^{p} P_m^i \left(u_{mjk}^{\tau}\Gamma_{mjk}^{\tau} - \theta_0^{\tau} u_{mjk}^0 \Gamma_{ijk}^0\right) + \frac{1}{L_2}\sum_{n=1}^{q} Q_n^j \left(v_{ink}^{\tau}\Gamma_{ink}^{\tau} - \theta_0^{\tau} v_{ink}^0 \Gamma_{ink}^0\right) +$$

$$+\sum_{s=1}^{h} S_s^k \left(\omega_{ijs}^{\tau}\Gamma_{ijs}^{\tau} - \theta_0^{\tau}\omega_{ijs}^0\Gamma_{ijs}^0\right) + \left(c^2\omega\right)_{ijk}^{\tau} - \theta_0^{\tau}\left(c^2\omega\right)_{ijk}^0 + \frac{1}{T}\sum_{\lambda=1}^{\vartheta}\theta_{\lambda}^{\tau}\left(\Gamma_{ijk}^{\lambda} - \Gamma_{ijk}^0\right) = 0 \tag{18}$$

$$(\tau = 1, 2, \dots, \vartheta).$$

In equations (12) and (14), it is sufficient to place the mark $\tau$ on the functions $u, v, \omega, \Phi, \Gamma$.

Thus, the problem of integrating system (1)–(4) has been reduced to the problem of solving a system of algebraic equations for determining $u_{ijk}^{\tau}$, $v_{ijk}^{\tau}$, $\omega_{ijk}^{\tau}$, $\Phi_{ijk}^{\tau}$, after $u_{ijk}^0$, $v_{ijk}^0$, $\omega_{ijk}^0$, $\Phi_{ijk}^0$ are known. Note that $\omega_{ijk}^0$ must be consistent with $u_{ijk}^0$, $v_{ijk}^0$ by means of relation (12), which is valid for any instants of time.

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[2] A. A. Dorodnitsyn, Proceedings of the Third All-Union Mathematical Congress, 2, 1956.

*Note: Figure translations are in progress. See original paper for figures.*