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Abstract

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ON THE THEORY OF BOUNDARY-VALUE PROBLEMS IN DOMAINS WITH DEGENERATE CONTOUR FOR VARIATIONAL AND DIFFERENTIAL EQUATIONS WITH PARTIAL DERIVATIVES

(Presented by Academician S. L. Sobolev on 20 VII 1961)

Boundary-value problems for variational and differential equations are considered in this paper in domains D of the space of n dimensions $x_1, x_2, \dots, x_n$. The boundary S of these domains is assumed to have a degenerate contour, i.e.

$$S = \sum_{s=1}^{n} S_{n-s};$$

here D and S are as in the papers ($^{1-7}$).

S. L. Sobolev was the first to formulate and solve the first (basic) boundary-value problem of this kind for polyharmonic equations $\Delta^m(u) = 0$. It was established that this problem can be solved by the variational method and, moreover, uniquely in the class of functions W_2^m, under a definite regularity in the boundary conditions. S. L. Sobolev showed that, in order to solve the indicated problem on the boundary domains of D, the manifolds S_{n-s}, it is possible to prescribe both the unknown function and all its derivatives up to order $(m - [s/2] - 1)$, inclusive*.

Subsequently, it was established by the author of the present paper (^{3}) that in the case of nonlinear boundary-value problems considered in the space W_p^m $(p > 1)$, for example in solving boundary-value problems for variational and differential equations connected with functionals (in the simplest case) of the form

$$D_m^p(u) = \int_D \cdots \int \sum_{\sum \alpha = m} C_{\alpha_1 \dots \alpha_n} \left| \frac{\partial^m u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \right|^p d\omega, \tag{1}$$

it proved possible to prescribe on the boundary of the domain D, on the manifolds S_{n-s}, both the unknown function and its derivatives already up to order

$(m - [s/p] - 1)$, inclusive. In this direction one should note the work of A. L. Krylov (^{24})**. For a more general form of the problem, see (^{4}).

Investigation of the question of boundary conditions for boundary-value problems of variational and differential equations of a very general type leads to the study of various closed sets contained in the spaces W_p^m. In the present paper new results in this direction are presented. The author previously introduced (^{4}) the functional spaces

$$W^{m_1 \dots m_k}_{p_1 \cdots p_k} \left\{ p_i \geq 1;\ m_{i+j} > m_i;\ j > 0:\ p_i > 1 : \left(\frac{1}{p_{i+j}} - \frac{m_{i+j} - m_i}{n} \right) \right\}, \qquad (2)$$

consisting of functions defined in D and having in it generalized (in the sense of S. L. Sobolev) derivatives up to order m_k, inclusive;

* All problems discussed in the present paper were considered under the assumptions customary for "direct variational methods" in the theory of boundary-value problems; moreover, the boundary conditions of the required function are taken, generally speaking, on the average with exponents corresponding to the embedding theorems for those spaces W in which these boundary-value problems are considered.

** This problem was formulated and solved by the author for generalized variational equations. A. L. Krylov indicated a certain class of problems in W_p^1 $(m = 1)$ for which transition to the Euler equation proved possible. Recently, transition to Euler equations has also proved possible for problems of a more general form.

$m_1, m_2, \dots, m_k$ are the orders of the derivatives of the functions of these spaces, which are summable in the domain D, respectively, with powers $p_1, p_2, \dots, p_k$.

$W^{m_1 \dots m_k}_{p_1 \cdots p_k} = W^{m_1}_{p_1} \cdot W^{m_2}_{p_2} \dots W^{m_k}_{p_k}$. In them one considered variational and differential equations generated, in the simplest case, by functionals of the type

$$D^{p_1 \dots p_k}_{m_1 \dots m_k}(u) = \int_D \cdots \int \sum_{i=1}^{k} \sum_{l=0}^{m_i} \sum_{\sum \alpha = l} C_{\alpha_1 \dots \alpha_n} \left| \frac{\partial^l u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \right|^p d\omega. \qquad (3)$$

The study of the properties of functions of these spaces* leads to the following:

Proposition 1. *The first boundary-value problem for the equations just indicated is solvable if, on the boundary manifolds S_{n-s} of the domain D, both the unknown function u itself and all its derivatives up to order*

$$\max(m_i - [s/p_i] - 1)$$

inclusive $(i = 1, 2, \dots, k)$ *are prescribed. In this case the variational problem has a unique solution.*

Of course, these results extend to functionals of a more general form than those given here, for example as in (4), where boundary-value problems with various boundary conditions are also considered. We note that, in the latter problems, the prescription of boundary conditions revealed possibilities different from those described earlier. Let us explain this by an example.

Let

$$W_{5,2}^{1,2}(D_4) = W_5^1(D_4) \cdot W_2^2(D_4)$$

be given (a space of 4 dimensions). In the case of the variational problem corresponding to the spaces W_2^2, W_5^1, $W_{5,2}^{1,2}$, on the boundary manifolds of the domain D it is possible to prescribe: 1) for W_2^2: on the manifold S_3, the unknown function and its first derivatives; on S_{4-s} $(s = 2, 3)$, only the unknown function; 2) for W_5^1: on S_{4-s} $(s = 1, 2, 3, 4)$, the unknown function is prescribed; 3) for $W_{5,2}^{1,2}$: on S_3 (as in W_2^2) the unknown function itself and its first derivatives are prescribed; on S_{4-s} $(s = 1, 2, 3, 4)$ it is possible to prescribe the unknown function (unlike the case W_2^2, here s may be equal to 4).

All this shows that the diversity in the boundary conditions of boundary-value problems for variational and differential equations is connected with those function spaces in which they are considered. The results of the investigations cited were based on the properties of functions of the spaces W_p^m (1–6).

Recently, in connection with the numerous applications of the theory of functions of real variables to mathematical physics, there has arisen, as is well known, a special interest in the so-called "weighted" classes of functions of many variables and in the equations associated with them, which "degenerate" on the boundary of the domain where the latter are considered. Investigations by various authors devoted to these questions, and the features of the present work, will be discussed at the end of the article.

In paper (7) the principal investigations on the theory of "weighted classes" were indicated: the author' s work for a negative exponent of the "weight" (2), the first systematic study of these classes by L. D. Kudryavtsev for positive exponents of the "weight," carried out in terms of the H-classes of S. N. Nikol' skii (mainly direct and inverse embedding theorems). One should also mention the works of P. I. Lizorkin, A. A. Vasharin, S. V. Uspenskii, and O. V. Besov (embedding theorems; boundary functions are subject to special conditions) $(8, 17, 19, 20, 22, 23, 25)$.

Continuing to assume, as before, that the domain D has a degenerate boundary, it is natural first of all to consider spaces of the form $W_{p,b_1\ldots b_n}^m$ (the definition of the space is given in the author' s paper (7)). The related–

* In them a metric can be introduced in accordance with (4) $(C_{\alpha_1\ldots\alpha_n} = 1)$ or analogously as for W_{pB}^m (5). The spaces are complete, and in them the corresponding "embedding" theorems and "compactness" theorems have been established.

for spaces of the simplest kind the functionals are as follows:

$$D^{m}_{p_1,b_1,\dots,b_n}(u) = \int_D \cdots \int \left\{ \sum_{k=0}^{m-1} \sum_{\Sigma\alpha=k} C_{\alpha_1\dots\alpha_n} \left| \frac{\partial^k u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right|^p + \prod_{s=1}^{n} r_{n-s}^{b_s} \sum_{\Sigma\alpha=m} C_{\alpha_1\dots\alpha_n} \left| \frac{\partial^m u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right|^p \right\} d\omega, \tag{4}$$

(the corresponding Euler equation I regard as known).

Introducing into the space under consideration a metric in accordance with (4), or as in paper (^{5}), we obtain that $W^m_{p,b_1\dots b_n}$ is a complete space, and the corresponding embedding and compactness theorems hold (see (^{7})).

The results of the investigation extend to functionals of a more general form, for example, as in (^{4}). In the paper it is shown that, when solving the corresponding variational and boundary-value problems in the space $W^m_{p,b_1\dots b_n}$, on the boundary manifolds S_{n-s} of the domain D one may prescribe both the unknown function and its derivatives up to order $m - [(s+b_s)/p] - 1$, inclusive.

By the same methods as in the case of ordinary (non- "weighted" spaces) it is shown that the corresponding variational problem for the functional (4) has a unique solution in $W^m_{p,b_1\dots b_n}$. At the same time the boundary-value problem for the variational equation occurring here is also solved (the generalized solution of the boundary-value problem for the Euler equation). For a broad class of problems a transition is effected from variational equations to the corresponding Euler equations. The boundary conditions are satisfied by the solution of the problems, generally speaking, on average with exponents that are clear from theorem A_5 in (^{7}). In particular, the boundary-value problem in domains with degenerate boundary is solved also for generalized polyharmonic equations; namely, the following holds:

Proposition 2. Under the usual assumptions for variational methods, the equation

$$\Delta^m(u)_{b_1\dots b_n} = \sum_{\Sigma\alpha=m} \frac{\partial^m}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \left(C_{\alpha_1\dots\alpha_n} \prod_{s=1}^{n} r_{n-s}^{b_s} \frac{\partial^m u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \right) = 0 \tag{5}$$

with boundary conditions

$$u\Big|_{S_{n-s}} = \varphi_0, \dots, 0, \dots, \frac{\partial^{m-[(s+b_s)/2]-1} u}{\partial x_n^{m-[(s+b_s)/2]-1}} = \varphi_{0,\dots,m-[(s-b_s)/2]-1} \tag{6}$$

has a solution in $W^{2m}_{,b_1\dots b_n}$.*

The solution of the problem satisfies the boundary conditions, generally speaking, on average with certain exponents q (in accordance with the embedding theorem stated earlier). Starting from the spaces W_p^m, the author earlier considered such intersections of these spaces in which the functions belonging to them possessed new properties in comparison with each class $W_{p_i}^{m_i}$ participating in the intersection. In the study of the boundary conditions of boundary-value problems for equations of a sufficiently general type that "degenerate" on the boundary of the domain D, here too a space is considered which is the intersection of the spaces $W_{p_i,b_1\ldots b_n}^{m_i}$, $i=1,2,\ldots,k$. The newly obtained space

$$W_{p_1\ldots p_k,b_1\ldots b_n}^{m_1\ldots m_k}$$

is such that the functions belonging to it possess additional properties relative to the properties of func-

* To this boundary-value problem there corresponds the variational problem on the minimum of the functional (4) with $C_{\alpha_1\ldots\alpha_n}^{\alpha}=0$ (for $p=2$) in the class of functions belonging to the space $W_{2,b_1\ldots b_n}^{m}$ (here also $p=2$) and satisfying the boundary conditions (6) in the sense indicated in the text.

⋯spaces participating in the intersection. In considering the corresponding variational and boundary-value problems, it turned out that on the boundary manifolds S_{n-s} of the domain D it is possible to prescribe both the sought function and all its derivatives up to order $\max(m_i-[(s+b_s)/p_i]-1)$, inclusive. The functionals investigated here are of the same type as in the author' s paper (4), but with the corresponding "degeneracy" factors in the integrand expressions. In the general case, the existence of solutions and the uniqueness of solutions of the boundary-value problem for the corresponding variational equation are established. For a sufficiently broad class of problems, a passage to the corresponding Euler equations is possible. The investigation is based on the properties of the function spaces $W_{p,b_1\ldots b_n}^{m}$.

Numerous investigations (8–21) have been devoted to the study of equations of type (5) (a domain without a degenerate contour, degeneration at the boundary of $n-1$ dimensions for $m=2$), both from the standpoint of equations of mixed type and from the standpoint of equations of elliptic type (mainly in the upper half-plane). In the present work, boundary-value problems are considered for the first time both for linear and for nonlinear "degenerating" equations of various orders. It should be emphasized that the domain D in which the indicated problems are solved has a "degenerate" contour

$$S=\sum_{s=1}^{n} S_{n-s},$$

and the degeneration is allowed both on all S_{n-s} $(s=1,2,\ldots,n)$ and on a part of the latter.

Variational boundary-value problems are considered not only in spaces, but also in their intersections. The case

$$W^{m_1\, m_2 \dots m_k}_{p_1\, p_2 \dots p_k} = W^{m_1}_{p_1} \cdot W^{m_2}_{p_2} \cdots W^{m_k}_{p_k}$$

(the intersection of the spaces of S. L. Sobolev) is singled out specially. At the same time, for the indicated problems a definite regularity is established in the prescription of boundary conditions, depending on the spaces in which they are considered. The properties of the spaces $W^m_{p_1, b_1 \dots b_n}$ have found application not only in the theory of boundary-value problems for variational equations, but also in the theory of systems of hyperbolic equations.

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Note: Figure translations are in progress. See original paper for figures.

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