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Abstract

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A. MISHCHENKO

ON FINALLY COMPACT SPACES

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§ 1. As is known, a topological space X is called finally compact in the sense of covers, or simply finally compact, if into each of its (open) covers ω there is inscribed a countable (or finite) cover ω', and without loss of generality one may assume that ω' is a subcover of the cover ω (i.e. $\omega' \subset \omega$). Already in the fundamental memoir of P. S. Aleksandrov and P. S. Uryson ([1]) the general notion of "compactness in a given interval of cardinalities" was introduced, leading, in particular, to the question of the equivalence of the just-formulated notion of final compactness and "final compactness in the sense of points of complete accumulation," which consists in the requirement that every uncountable set M have in the given space X a point of complete accumulation (i.e. such a point x_0 every neighborhood of which meets the set M in a set of the same cardinality as the whole set M). P. S. Aleksandrov and P. S. Uryson proved there in ([1]) that final compactness in the sense of points of complete accumulation means that every cover ω of uncountable regular cardinality contains a cover of cardinality smaller than the cardinality of ω (whence it follows that every finally compact space (in the sense of covers) is also finally compact in the sense of points of complete accumulation). The main result of the present work is an example (§ 3) of a completely regular space, finally compact in the sense of points of complete accumulation, but not finally compact (in the sense of covers).

I do not know whether this space is normal. We shall see in § 4 that any example of a normal space finally compact in the sense of points of complete accumulation and not finally compact (in the sense of covers) would at the same time be an example of a normal non-countably paracompact space, i.e. would solve Dowker' s well-known problem ([2]).

§ 2. Denote by $TW(\omega_k)$ the topological space of all ordinal numbers $\alpha < \omega_k$. If R_α are ordered sets, then $\prod_\alpha R_\alpha$ will be considered partially ordered according to the rule: if $x = \{x_\alpha\} \in \prod_\alpha R_\alpha$, $y = \{y_\alpha\} \in \prod_\alpha R_\alpha$, then $x < y$ if and only if $x_\alpha < y_\alpha$ for all indices α. We preface the construction of the example with several lemmas.

Lemma 1. The space $TW(\omega_k)$ is $[\aleph_0, \aleph_{k-1}]$-compact.

Proof. Let π be a cover of cardinality $\leq \aleph_{k-1}$. It is enough to prove that among the elements Γ of the cover π there is a $\Gamma_0 \in \pi$ containing an "infinite interval" of the form $(\alpha_0, \infty) = \{\alpha : \alpha > \alpha_0\}$. But for every $\alpha < \omega_k$, obviously, there is such a $\beta(\alpha) < \alpha$ that the half-interval $(\beta(\alpha), \alpha]$ is contained in some $\Gamma \in \pi$. Then there exist an $\alpha_0 < \omega_k$ and an $M \subseteq W(\omega_k)$, cofinal in the whole of $W(\omega_k)$, such that for all $\alpha \in M$ we have $\beta(\alpha) < \alpha_0 < \alpha$. Since $\operatorname{card} M = \aleph_k$, and $\operatorname{card} \pi \leq \aleph_{k-1}$, there exists an $M' \subseteq M$ of the same cardinality $\aleph_k$ such that for all $\alpha \in M'$ the half-intervals $(\beta(\alpha), \alpha]$ lie in one and the same element $\Gamma_0 \in \pi$. But then also $(\alpha_0, \infty) \subseteq \Gamma_0$, as was required to prove.

Lemma 2. $R = \prod_{i=k}^{n} TW(\omega_i)$ is an $[\aleph_0, \aleph_{k-1}]$-compact space.

Proof by induction. For $n = k$ the assertion is true by Lemma 1. Suppose the assertion is true for $n-1$. Let π be a cover of R of cardinality less than $\aleph_k$. Denote by R_1 the set $\prod_{i=k}^{n-k} TW(\omega_i)$, and by R_x the set $\{x\} \times TW(\omega_n)$, where $x \in R_1$. For any point $p \in R_x$, $p = (x, \alpha)$, there exist $x(\alpha) \in R_1$, $\beta(\alpha) < \omega_n$, $\Gamma_\alpha \in \pi$, such that $(q(\alpha), p] \subseteq \Gamma_\alpha$, where $q(\alpha)$ is the point $(x(\alpha), \beta(\alpha))$. Then there exists a set M, cofinal in $W(\omega_k)$, such that $\beta(\alpha)$ is less than some fixed $\alpha_0 < \omega_n$ for all $\alpha \in M$. The set S of all pairs (x, Γ), $x \in R_1$, $\Gamma \in \pi$, has cardinality $\aleph_{n-1}$. Therefore there exists $M' \subseteq M$ of the same cardinality $\aleph_{n-1}$, cofinal in $W(\omega_n)$, such that for any $\alpha \in M'$ we have $x(\alpha) \equiv x_0$, $\Gamma_\alpha \equiv \Gamma_0$, i.e. $(p_0, r_0) \subset \Gamma_0$, where $p_0 = (x_0, \alpha_0)$, $r_0 = (x, \infty)$. Obviously, Γ_0, x_0, α_0 are functions of the point $x \in R_1$: $\alpha_0 = \alpha_0(x)$, $x_0 = x_0(x)$, $\Gamma_0 = \Gamma_0(x)$. Since the cardinality of R_1 is $\aleph_{n-1}$, there exists $\bar{\alpha} > \alpha_0(x)$ for any $x \in R_1$. Then, putting $\bar{p}_0(x) = (x_0(x), \bar{\alpha})$, we have $(\bar{p}_0(x), \bar{r}_0(x)) \subset \Gamma_0(x)$. Put

$$\xi(\Gamma) = \bigcup_{\Gamma(x)=\Gamma} (x_0(x), x).$$

The set $\{\xi(\Gamma)\}_{\Gamma \in \pi}$ covers R_1 and, by the induction hypothesis, has a finite subcover $\{\xi(\Gamma_1), \dots, \xi(\Gamma_s)\}$. Then the system of elements $\{\Gamma_1, \dots, \Gamma_s\}$ covers the set $R_1 \times (\bar{\alpha}, \infty)$. The space R can be represented in the form:

$$R = R_1 \times [1, \bar{\alpha}] \cup R_1 \times (\bar{\alpha}, \infty).$$

Since $[1, \bar{\alpha}]$ is bicompact, $R_1 \times [1, \bar{\alpha}]$ is $[\aleph, \aleph_{k-1}]$-compact, i.e. it is covered by a finite number of elements $\{\Gamma_{s+1}, \dots, \Gamma_m\}$. The system $\{\Gamma_1, \dots, \Gamma_m\}$ covers the space R.

Lemma 3. Let

$$R = [1, \beta] \times \prod_{i=k}^{s} TW(\omega_i).$$

Further, denote by $\operatorname{cf} \beta$ the least ordinal cofinal with β. Suppose that $\operatorname{cf} \beta < \omega_k$. Finally, let

$$G \supset \{\beta\} \times \prod_{i=k}^{s} TW(\omega_i)$$

be some set.

Then there exists $\alpha < \beta$ such that

$$G \supset (\alpha, \beta] \times \prod_{i=k}^{s} TW(\omega_i).$$

Proof. For $s = k$ the proof is clear. Suppose the lemma is true for $s-1$. Denote by R^* the set

$$[1, \beta] \times \prod_{i=1}^{s-1} TW(\omega_i),$$

and by R'^* the set

$$\{\beta\} \times \prod_{i=1}^{s-1} TW(\omega_i).$$

For any $y \in R'^*$ and any $\alpha \in W(\omega_s)$ there exist $z(y, \alpha) \in R^*$, $\beta(y, \alpha) \in W(\omega_s)$, such that

$$(z(y, \alpha), y] \times (\beta(y, \alpha), \alpha] \subset G.$$

Then there exists $z(y)$ such that

$$(z(y), y] \times TW(\omega_s) \subset G.$$

$\bigcup_{y \in R'^*} (z(y), y]$ is an open set in R^*. By the induction hypothesis there exists α such that

$$[\alpha, \beta] \times \prod_{i=1}^{s-1} TW(\omega_i) \subseteq \bigcup_{y \in R'^*} (z(y), y].$$

Then

$$(\alpha, \beta] \times \prod_{i=1}^{s} TW(\omega_i) \subset G.$$

Lemma 4. The space

$$R = \prod_{i=k}^{\infty} TW(\omega_i)$$

is $[\aleph_0, \aleph_{k-1}]$-compact.

Proof. Let π be a countable cover having no finite subcover and such that

$$\pi = \{O_n\}_{n=k}^{\infty}, \qquad O_n = \Gamma_n \times \prod_{i=n+1}^{\infty} TW(\omega_i),$$

where $\Gamma_n \subset \prod_{i=k}^{n} TW(\omega_i)$. Suppose that for every $\alpha \in W(\omega_k)$ the set

$$\{\alpha\} \times \prod_{i=k+1}^{\infty} TW(\omega_i)$$

is covered by a finite number of elements of the cover π, i.e., that there exists $N(\alpha)$ such that

$$\bigcup_{n=k}^{N(\alpha)} O_n \supset \{\alpha\} \times \prod_{i=k+1}^{\infty} TW(\omega_i).$$

By Lemma 3 there exists $\beta(\alpha)$ such that

$$\bigcup_{n=k}^{N(\alpha)} O_n \supset (\beta(\alpha), \alpha] \times \prod_{i=k+1}^{\infty} TW(\omega_i).$$

Let

$$\tau(N) = \bigcup_{N=N(\alpha)} (\beta(\alpha), \alpha].$$

$\{\tau(N)\}_{N=k}^{\infty}$ covers the space $TW(\omega_k)$, and there exists N_0 such that

$$\bigcup_{N=k}^{N_0} \tau(N) \supset TW(\omega_k).$$

Then

$$\bigcup_{n=k}^{N_0} O_n \supset R.$$

The contradiction shows that there exists an α_k such that the set

$$\{\alpha_k\} \times \prod_{i=k+1}^{\infty} TW(\omega_i)$$

is not covered by a finite number of elements of the cover π.

In an entirely analogous way it is proved that if

$$\{\alpha_k\} \times \dots \times \{\alpha_n\} \times \prod_{i=n+1}^{\infty} TW(\omega_i)$$

is not covered by a finite number of elements of the cover π, then there exists an α_{n+1} such that

$$\{\alpha_k\} \times \dots \times \{\alpha_n\} \times \{\alpha_{n+1}\} \times \prod_{i=n+2}^{\infty} TW(\omega_i)$$

is not covered by a finite number of elements of the cover π. It is easy to see that the set

$$\{\alpha_k\} \times \dots \times \{\alpha_n\} \times \prod_{i=n+1}^{\infty} TW(\omega_i)$$

does not intersect $O_k, \dots, O_n$. Then the point $\{\alpha_i\}_{i=k}^{\infty}$ is not contained in any of the elements $O_i \in \pi$, which refutes our initial assumption. Let now π be an arbitrary cover of cardinality less than $\aleph_k$. Every element O of the cover π can be represented in the form:

$$O = \bigcup_{i=k}^{\infty} \left(G_{iO} \times \prod_{j=i+1}^{\infty} TW(\omega_j) \right), \quad \text{where} \quad G_{iO} \subset \prod_{j=k}^{i} TW(\omega_j).$$

Let

$$O_i = \bigcup_{O \in \pi} G_{iO} \times \prod_{j=i+1}^{\infty} TW(\omega_j).$$

$\{O_i\}_{i=k}^{\infty}$ covers the space R. Then there exists an N such that

$$\bigcup_{i=k}^{N} O_i \supset R.$$

The system of sets

$$\left\{ G_{iO} \times \prod_{j=i+1}^{N} TW(\omega_j) \right\}_{i=1,\, O \in \pi}^{N}$$

covers the space

$$\prod_{i=k}^{N} TW(\omega_i).$$

By Lemma 2 there exists a finite subcover. Hence π contains a finite subcover.

§ 3. We pass to the construction of the principal example. Let

$$R = \prod_{i=1}^{\infty} TW(\omega_i + 1), \qquad R_k = \prod_{i=1}^{k} TW(\omega_i + 1) \times \prod_{i=k+1}^{\infty} TW(\omega_i), \qquad R_k \subset R.$$

The desired space is the subspace $R^* \subset R$:

$$R^* = \bigcup_{k=1}^{\infty} R_k.$$

Since R_i is $(\aleph_0, \aleph_{i-1})$-compact and

$$R^* = \bigcup_{k=n}^{\infty} R_k$$

for every number n, it follows that R^* is $(\aleph_1, \aleph_{\omega_0})$-compact in the sense of covers. Since R^* has a ba-

of cardinality $\aleph_\omega$, then R^* is $[\aleph_{\omega+1}, \infty)$-compact in the sense of coverings. We show that there exists a covering π of cardinality $\aleph_\omega$ which has no subcovering of smaller cardinality. The desired covering is the union of a countable set of systems π_i; $\pi_i = \{\Gamma_{\alpha,i}\}_{\alpha<\omega_i}$, $\Gamma_{\alpha,i} = \{x : \ x \in R^*, \ x = \{x_k\}, \ x_i < \alpha\}$. The system

$$\pi = \bigcup_{i=1}^{\infty} \pi_i$$

is a covering. Suppose that there exists a subcovering of smaller cardinality. Then there exists a countable subcovering

$$\pi^* = \bigcup_{i=1}^{\infty} \pi_i^*.$$

We may assume that π_i^* consists of one element $\Gamma_{\alpha_i,i} \in \pi_i$. Then the point

$$x = \{\alpha_i + 1\}_{i=1}^{\infty}$$

does not belong to any of the elements $\Gamma_{\alpha_i,i} \in \pi_i^*$. We have arrived at a contradiction!

§ 4. We give some sufficient conditions under which the two definitions of final compactness are equivalent.

Theorem 1. *Let the space R be normal, countably paracompact, and finally compact in the sense of complete accumulation points. Then R is finally compact in the sense of coverings.*

Proof. Suppose the contrary; let π be a covering of least cardinality m which has no countable subcovering. Then, by a result of A. S. Aleksandrov and P. S. Uryson, the cardinality of π is irregular and

$$\pi = \bigcup_{i=1}^{\infty} \pi_i,$$

where the cardinality of π_i is less than the cardinality of π. Denote by $\tilde{\pi}_i$ the "body of the system π_i," i.e., the union of all sets forming this system; $\{\tilde{\pi}_i\}$ covers R. Since R is normal and countably paracompact, there exist closed sets $F_i \subset \tilde{\pi}_i$ which also cover R. Further, there exists a countable system $\pi_i^* \subseteq \pi_i$ covering the closed set F_i (this follows from the fact that the covering of the space R consisting of the elements of the system π_i and of the set $R \setminus F_i$ has cardinality $< m$). The system

$$\bigcup_{i=1}^{\infty} \pi_i^*$$

is countable and covers R. The theorem is proved.

Sufficient conditions for the equivalence of $[a, b]$-compactness in the sense of coverings and in the sense of complete accumulation points are formulated analogously. For example, the following condition will be such a sufficient condition.

A. For every open covering $\{\Gamma_\alpha\}$ of the space there exists a closed covering $\{F_\alpha\}$ combinatorially inscribed in $\{\Gamma_\alpha\}$ (i.e., $F_\alpha \subseteq \Gamma_\alpha$).

For hereditary $[a, b]$-compactness we have

Theorem 2. *Let a be a regular cardinal, and let R be a hereditarily $[a, b]$-compact space in the sense of complete accumulation points (no separation axioms are assumed). Then R is hereditarily $[a, b]$-compact (in the sense of coverings).*

Proof. Let m be an irregular cardinal; denote by $\chi(m)$ the least cardinal such that m can be represented as a sum of cardinals $< m$ taken in number $\chi(m)$. Let π denote an arbitrary system of open sets of irregular cardinality m, where $a < m \leq b$, $\chi(m) < a$. It is enough to prove that π contains a subsystem of cardinality $< m$ with the same body. Suppose the contrary; let π be a system of least cardinality having no subsystem of smaller cardinality with the same body. Then

$$\pi = \bigcup_{\alpha \in A} \pi_\alpha,$$

where card $A < a$, and card $\pi_\alpha <$ card π. Since the property of $[a, b]$-compactness in the sense of complete accumulation points is hereditary and card $\pi_\alpha <$ card π, there exists a subsystem $\pi^*_\alpha \subseteq \pi_\alpha$ with the same body and card $\pi^*_\alpha < a$. Then

$$\operatorname{card} \bigcup_{\alpha \in A} \pi^*_\alpha \leq a,$$

and $\bigcup_{\alpha \in A} \pi^*_\alpha$ covers the body of π. Hence the assumption is false and the theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.