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Abstract

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ON THE THEORY OF NORMAL DYNAMICAL SYSTEMS

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§ 1. **Introduction.** Let $s = \{\omega = \{\omega_k\}_{k=-\infty}^{\infty}\}$ be the space of all two-sided infinite real sequences with the usual (weak) topology; let $s^* = \{x = \{x_k\}_{k=-\infty}^{\infty}\}$ be the conjugate space of finite sequences; $T(s \to s)$ the linear shift operator on s: $T\{\omega_k\} = \{\omega_{k+1}\}$. A measure μ in s, defined on the σ-algebra of Borel sets, is called Gaussian if every linear functional $x \in s^*$ on s has a Gaussian distribution.

The aggregate $\{s, \mu, T\}$ is called a **one-dimensional normal dynamical system** (n.d.s.) if the following conditions are satisfied: a) $\int_s \omega_k \, d\mu = 0$; b) $\int_s \omega_k^2 \, d\mu = 1$, $k = 0, \pm 1, ...$; c) $\mu(TA) = \mu(A)$, where A is a Borel set in s. A one-dimensional n.d.s. is completely determined by specifying the bilinear correlation functional on s^*:

$$B(x, y) = \int_s x(\omega) y(\omega) \, d\mu; \quad x, y \in s^*,$$

i.e. by an infinite matrix $\{\tilde{b}_{k_1,k_2}\}$,

$$\tilde{b}_{k_1 k_2} = \int_s \omega_{k_1} \omega_{k_2} \, d\mu.$$

From conditions b) and c) it follows that: $\tilde{b}_{k_1,k_1} = 1$, $\tilde{b}_{k_1,k_2} = \tilde{b}_{k_1-k_2,0}$. Let $\tilde{b}_{k,0} = b_k$; the sequence $\{b_k\}$ is positive definite and is called the correlation sequence of the n.d.s. By a well-known theorem, there exists a symmetric measure on $[-\pi, \pi]$, $F(d\lambda)$, called the spectral measure of the n.d.s., such that

$$b_k = \int_{-\pi}^{\pi} e^{ik\lambda} F(d\lambda).$$

§ 2. **The unitary ring $\mathcal{L}_G$.** Consider on $[-\pi, \pi]$ an arbitrary symmetric measure $G(d\lambda)$. The measure $G^{(n)} = \underbrace{G \times \cdots \times G}_{n}$ is defined on the product of n intervals $[-\pi, \pi]$. Denote by $L^2_{G(n)}$ the space of complex-valued functions, symmetric in all arguments and square-summable with respect to the measure $G^{(n)}$; $L^2_{G(0)} = R^1$. Introduce the Hilbert space

$$\mathcal{L}_G = \sum_{n=0}^{\infty} \oplus L^2_{G(n)}.$$

Let $f_n \in L^2_{G(n)}$, $g_m \in L^2_{G(m)}$; define the product of these functions by the formula

$$f_n \times g_m = \sum_{k=0}^{\min(n,m)} q_k^{(n,m)}\, S\left[\underbrace{\int \cdots \int}_{k} f_n(\alpha_1, \dots, \alpha_k, \lambda_1, \dots, \lambda_{n-k}) \times \right.$$

$$\left. \times\, g_m(-\alpha_1, \dots, -\alpha_k, \theta_1, \dots, \theta_{m-k}) G(d\alpha_1) \cdots G(d\alpha_k)\right], \tag{1}$$

$$q_k^{(n,m)} = \sqrt{\frac{n!\, m!}{(m+n-2k)!\,(n-k)!\,(m-k)!\,k!}};$$

$$S[h(\lambda_1, \dots, \lambda_r)] = \sum_{(i_1 \dots i_r)} h(\lambda_{i_1}, \dots, \lambda_{i_r}).$$

Define the involution for $f_n \in L^2_{G(n)}$ by the formula

$$[f_n(\lambda_1, \dots, \lambda_n)]^* = \overline{f_n(-\lambda_1, \dots, -\lambda_n)}. \tag{2}$$

Introduce the operator V on $L^2_{G(n)}$

$$(V f_n)(\lambda_1, \dots, \lambda_n) = \exp\left[i(\lambda_1 + \dots + \lambda_n)\right] f_n(\lambda_1, \dots, \lambda_n). \tag{3}$$

It is not difficult to verify that the natural extension of the operations (1), (2), (3) to $\mathcal{L}_G$ turns $\mathcal{L}_G$ into a unitary ring in the sense of (1), and V into a multiplicative unitary operator, i.e. into an automorphism of the ring $\mathcal{L}_G$.

Let now $\{s, \mu, T\}$ be a one-dimensional n.d.s. Consider the Hilbert space of all complex-valued functionals on s that are summable with the square of the modulus with respect to the measure μ. Denote it by $L^2_\mu(s)$. To the shift T on s there corresponds in $L^2_\mu(s)$ the operator U_T: $(U_{TX})(\omega) = X(T\omega)$, which is an automorphism of the unitary ring $L^2_\mu(s)$.

Theorem 1. *Let $\{s, \mu, T\}$ be a one-dimensional n.d.s. with spectral measure $F(d\lambda)$. The automorphism U_T of the unitary ring $L^2_\mu(s)$ is isomorphic to the automorphism V of the unitary ring $\mathcal{L}_F$.*

Proof. We shall call a functional

$$X_{k_1,\dots,k_n}(\omega) = \prod_{r=1}^{n} \omega_{k_r} - \sum_{(i,j)} b_{k_i-k_j} \prod_{r\neq i,j} \omega_{k_r} +$$

$$+ \sum_{(i,j),(l,p)} b_{k_i-k_j} b_{k_l-k_p} \prod_{r\neq i,j,l,p} \omega_{k_i} - \dots \tag{4}$$

a generalized Hermite polynomial of degree n.

(For the case R^1 and the ordinary Gaussian measure, formula (4) gives the ordinary Hermite polynomials; see, for example, ([2]).) Let H_n be the linear closed span of the generalized Hermite polynomials of degree n. It is not difficult to show that

$$L^2_\mu(s) = \sum_{n=0}^{\infty} \oplus H_n.$$

An isomorphism of the unitary rings $L^2_\mu(s)$ and $\mathcal{L}_F$ is given by the formula:

$$QX_{k_1\dots k_n} = \frac{1}{\sqrt{h!}} S\left[\exp\left[i\sum_{r=1}^{n} k_r\lambda_r\right]\right], \qquad \text{and then } QU_TQ^{-1} = V.$$

The spectral part of Theorem 1 was first established by Itô ([3]) with the aid of stochastic integrals. The method of polynomial expansion given above carries over to the case of spaces $L^2_\mu(E)$, where E is an arbitrary locally convex linear topological space and μ is a Gaussian measure in it. This method may be useful in various problems–approximation of functionals, continual integration, etc. For the particular case (“white noise”) it was considered in Wiener’ s book ([4]). It should be noted that the study of the automorphism V of the ring $\mathcal{L}$ is in many respects simpler than the study of the automorphism U_T of the ring $L^2_\mu(s)$.

§ 3. **The standard unitary ring.** Let F_0 be Lebesgue measure on $[-\pi,\pi]$. The unitary ring $\mathcal{L}_{F_0} = \mathcal{L}$ will be called **standard**. A measure μ in s for which $F = F_0$ is the direct countable product of a fixed Gaussian measure on R^1 by itself. Take an arbitrary continuous symmetric measure F on $[-\pi,\pi]$ and establish an isomorphism of the rings $\mathcal{L}_F$ and $\mathcal{L}$. For this choose some measurable, odd, one-to-one almost everywhere function $\varphi(\lambda)$ carrying the measure F_0 into F. Define the isomorphism W ($\mathcal{L}_F \to \mathcal{L}$):

$$Wg_n(\lambda_1,\dots,\lambda_n) = f_n(\lambda_1,\dots,\lambda_n) = g(\varphi(\lambda_1),\dots,\varphi(\lambda_n)); \quad g_n \in L^2_{F(n)},\ f_n \in L^2_{F_0(n)}.$$

The isometricity and multiplicativity of the operator W are verified directly. The automorphism V of the ring-

$\mathcal{L}_F$ passes into $V_\varphi = WVW^{-1}$:

$$(V_\varphi f_n)(\lambda_1, \dots, \lambda_n) = \exp\left[i \sum_{r=1}^{n} \varphi(\lambda_r)\right] f_n(\lambda_1, \dots, \lambda_n). \qquad (V_\varphi)$$

Let us note that the transformation W of the ring $\mathcal{L}_F$ onto $\mathcal{L}$ corresponds to a certain linear transformation P of the space s onto itself, under which the correlation matrix of the N.d.s. with spectral measure F is brought to diagonal form; to the automorphism V_φ in $\mathcal{L}$ there corresponds the operator PTP^{-1}, where T is the shift in s.

If the spectral measure F is not continuous (in this case the N.d.s. is nonergodic), then the isomorphism of $\mathcal{L}_F$ and $\mathcal{L}$ is effected with the aid of a function $\varphi(\lambda)$, which may be constant on some sets of positive Lebesgue measure. In particular, if $F = \delta_0$ (unit mass at zero), then $\varphi(\lambda) \equiv 0$ and $V_\varphi = E$, where E is the identity automorphism. As the following theorem shows, the arbitrariness in the choice of the function $\varphi(\lambda)$ is inessential.

Theorem 2. Let the measurable odd functions $\varphi(\lambda)$ and $\psi(\lambda)$ be metrically isomorphic, i.e. $\varphi(\lambda) = \psi(N\lambda)$, where N is a measurable, invertible, Lebesgue-measure-preserving transformation of $[-\pi, \pi]$ onto itself. Then $V_\psi = UV_\varphi U^{-1}$, where U is some automorphism of the ring $\mathcal{L}$.

Thus, the study of one-dimensional N.d.s. reduces to the study of automorphisms of the ring $\mathcal{L}$ of the form (V_φ), where $\varphi(\lambda)$ is a measurable odd function mapping $[-\pi, \pi]$ into itself and metrically isomorphic to some measurable odd monotone, in the broad sense, function $\psi(\lambda)$ that also maps $[-\pi, \pi]$ into itself.

§ 4. **Normal automorphisms and multidimensional N.d.s.** Within the framework of the unitary ring $\mathcal{L}$ it is natural to study a broader class of dynamical systems than one-dimensional N.d.s.

Definition. An automorphism V_φ of the ring $\mathcal{L}$ of the form (V_φ), where $\varphi(\lambda)$ is an arbitrary odd measurable function mapping $[-\pi, \pi]$ into itself, is called a **normal automorphism**. If the multiplicity of the continuous part of the spectrum of a normal automorphism V_φ in the subspace $L^2_{F_0^1}$ is finite and equal to n, then it is called n-dimensional; otherwise, infinite-dimensional.

Let us introduce multidimensional N.d.s. into consideration. By this we mean a collection $\{s, \mu, T\}$, where s is the space of sequences, infinite in both directions, of real vectors of an n-dimensional space, μ is a Gaussian measure in s, invariant with respect to the shift T. The case of infinite n is not excluded. A multidimensional N.d.s. is completely determined by the spectral matrix $\|F_{ij}(d\lambda)\|$ (see, for example, ([5])).

Theorem 3. Let $\{s, \mu, T\}$ be an n-dimensional N.d.s. The automorphism U_T of the unitary ring $L^2_\mu(s)$, corresponding to the shift T in s, is isomorphic to some normal automorphism of the ring $\mathcal{L}$ of dimension not greater than n.

§ 5. **Some properties of normal automorphisms.**

Theorem 4. The normal automorphisms V_φ of the ring $\mathcal{L}$ form a commutative subgroup of the group of all automorphisms. The uniform operator topology on this subgroup is discrete; weak convergence of one-dimensional normal automorphisms corresponding to a sequence of one-dimensional N.d.s. is equivalent to weak convergence of the spectral measures of these N.d.s.

A flow $V_{\varphi_\alpha}(\alpha)$ $(-\infty < \alpha < \infty)$ composed of normal automorphisms will be called a continuous normal flow if $(V_{\varphi_\alpha}(\alpha)f, f)$ is a continuous function of α for all $f \in \mathcal{L}$.

Theorem 5. Every normal automorphism V_φ can be uniquely embedded in a continuous normal flow:

$$V_{\varphi_\alpha}(\alpha) = V_{\alpha\varphi}.$$

Let us indicate some consequences of the preceding theorems.

1. If the matrix $\|F_{ij}\|$ of a multidimensional n.d.s. has mutually singular diagonal elements F_{ij}, then the corresponding n.d.s. is metrically isomorphic to a one-dimensional n.d.s. with spectral measure $F = \sum F_{ii}$.
2. The maximal spectral type of an n-dimensional n.d.s. with spectral matrix $\|F_{ij}\|$ is equal to $\sum_{k=0}^{\infty} \Phi^k$, where $\Phi = \sum F_{ii}$, $\Phi^0 = \delta_0$, and Φ^k is the power in the sense of convolution (a generalization of the result obtained by S. V. Fomin for $n = 1$ ([6])).
3. It is interesting to note that there exist n-dimensional n.d.s. (n finite) which are not isomorphic to any m-dimensional ones for $m < n$. Thus, if G is the spectral measure of an n.d.s. with simple spectrum (a description of such G is given in ([7])), then the n-dimensional n.d.s. with matrix $\|\delta_{ik}G\|$ ($\delta_{ik} = 0$ for $i \neq k$; $\delta_{ik} = 1$ for $i = k$) is not isomorphic to any m-dimensional n.d.s. for $m < n$, since an m-dimensional normal automorphism cannot have more than m cyclic subspaces with spectral type G. If one takes into account that powers of a normal automorphism with simple spectrum are normal automorphisms also with simple spectrum, then from the example just given it follows that the direct product of n copies of one and the same automorphism is not necessarily isomorphic to the nth power of the same automorphism. This same example gives a negative answer to the question posed by K. Ito (([8]), p. 117): is every stationary process some function of a one-dimensional Gaussian process? It remains an open question whether every stationary process is a function of some multidimensional Gaussian process.

4. Every n-dimensional ($n < \infty$) normal automorphism is isomorphic to the nth power of some one-dimensional normal automorphism, and conversely.
5. The entropy of an n-dimensional n.d.s. is equal to 0 or ∞, and the first case occurs if and only if all elements of the spectral matrix of the n.d.s. are singular with respect to Lebesgue measure. This result is easily derived from the corresponding fact for one-dimensional n.d.s., proved by M. S. Pinsker ([9]).

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Note: Figure translations are in progress. See original paper for figures.